Guarantees in Fair Division: general or monotone preferences

Anna Bogomolnaia♠, Hervé Moulin♠ and Richard Stong¶

November 2019

♠ University of Glasgow, UK, and Higher School of Economics, St Petersburg, Russia

¶ Center for Communications Research, San Diego, U.S.A.

Abstract

A basic test of fairness when we divide a "manna" Ω of private items between n agents is the lowest welfare the rule guarantees to each agent, irrespective of others' preferences. Two familiar examples are: the *Equal Split Guarantee* (the utility of $\frac{1}{n}\Omega$) when the manna is divisible and preferences are convex; and $\frac{1}{n}$-th of the utility of a heterogenous non atomic "cake", if utilities are additive.

The *minMax* utility of an agent is that of her best share in the worst possible n-partition of Ω. It is weakly below her *Maxmin* utility, that of her worst share in the best possible n-partition. The Maxmin guarantee is not feasible, even with two agents, if non convex preferences are allowed. The minMax guarantee is feasible in the very general class of problems where Ω is non atomic and utilities are continuous, but not necessarily additive or monotonic. The proof uses advanced algebraic topology techniques. And the minMax guarantee is implemented by the n-person version of Divide and Choose due to Kuhn (1967).

When utilities are co-monotone (a larger part of the manna is weakly better for everyone, or weakly worse for everyone) better guarantees than minMax are feasible. In our *Bid & Choose* rules, agents bid the smallest size (according to some benchmark measure of Ω) of a share they find acceptable, and the lowest bidder picks such a share. The resulting guarantee is between the minMax and Maxmin utilities.

Acknowledgement 1 *We are grateful for the critical comments of Haris Aziz, Steve Brams, Ariel Procaccia, Shahar Dobzinski, and participants in seminars and workshops at LUISS University, UNSW Sydney, Universities of Gothenburg, St Andrews, the London School of Economics, and the 12th International Symposium on Algorithmic Game Theory, Athens. Special thanks to Erel Segal-Halevi and Ron Hozman for their detailed comments on the first version of this paper.*

Bogomolnaia and Moulin acknowledge the support from the Basic Research Program of the National Research University Higher School of Economics.

1 Introduction and the punchlines

The fair division of a common property manna – resources privately consumed – between its joint owners is a complicated problem if we also wish to take efficiently into account differences in individual preferences. When information about preferences remains private, a coarse yet important benchmark attached to a particular division rule is the welfare *Guarantee* it offers to each participant. This is the highest welfare that a given agent can secure in this rule, irrespective of the preferences of other agents, even if our agent is clueless about the latter and assumes the worst. The more an agent is risk averse and the less she knows about others' preferences, the more this worst case measure matters to her.

Formally we assume that the manna Ω and the domain $\mathcal{D}$ of potential preferences are common knowledge, and define a *Fair Guarantee* as a mapping $(u_i, n) \to \Gamma(u_i; n)$ selecting for each preference in $\mathcal{D}$, described for clarity as a utility function u_i, and each number n of joint owners, a utility level. The mapping is fair because it ignores agent i's identity, and it should be feasible: for any profile $(u_i)_{i=1}^n$ of utilities in $\mathcal{D}^n$ there must exist a partition $(S_i)_{i=1}^n$ of Ω such that $u_i(S_i) \geq \Gamma(u_i; n)$ for all i.

The two far reaching questions are: what are the best (highest) Fair Guarantees in a given division problem $(\Omega, \mathcal{D})$? and what mechanism implements[1] them? Steinhaus' 1948 paper ([34]) that launched the modern fair division literature identifies the highest Fair Guarantee in the cake-cutting model with additive utilities, and proposes a rule to implement it. We pursue here this program when individual preferences vary in a (much) more general

[1]In the simple sense of implementation described in the last paragraph of this section.

domain.

Observe first that any Fair Guarantee $\Gamma(u;n)$ is bounded above by the utility of the worst share for u in the best n-partition of the manna. Writing this benchmark utility as $Maxmin(u;n)$, we have for any Γ,u in $\mathcal{D}$ and n

$$\Gamma(u;n) \leq Maxmin(u;n) = \max_{\Pi=(S_i)_{i=1}^n} \min_{1\leq i\leq n} u(S_i) \tag{1}$$

where the maximum (that may not be achieved exactly) bears on all n-partitions $\Pi = (S_i)_{i=1}^n$ of Ω. The claim follows at once from the feasibility of $\Gamma(u;n)$ at the unanimous profile where $u_i = u$ for all i: there is a partition Π such that $u(S_i) \geq \Gamma(u;n)$ for all i, hence $\Gamma(u;n) \leq \min_{1\leq i\leq n} u(S_i) \leq Maxmin(u;n)$.

Therefore if $(u,n) \to Maxmin(u;n)$ is itself a Fair Guarantee (it is fair, but the issue is feasibility), it is the best possible one and answers the first of the two general questions above. This happens in two well known and much discussed families of fair division problems.

Steinhaus assumes the manna Ω is a non atomic measurable space and $\mathcal{D}$ is the set of additive utilities non atomic with respect to the underlying measure. Additivity of u implies $Maxmin(u;n) \leq \frac{1}{n}u(\Omega)$; this is in fact an equality because by Lyapounov theorem the cake can be partitioned in n shares of equal utility; the same theorem implies also that the utility profile $(\frac{1}{n}u_i(\Omega))_{i=1}^n$ is feasible. The corresponding welfare lower bound is known as the *Proportional Guarantee*: $u_i(S_i) \geq \frac{1}{n}u_i(\Omega)$ where S_i is agent i's share. It is the weakest and omnipresent fairness requirement in the subsequent cake-cutting literature (surveyed in [13] and [31]).

Consider next the microeconomic model of fair division where the manna is a bundle $\omega \in \mathbb{R}_+^K$ of K divisible and non disposable items, and $\mathcal{D}$ is the set of convex and continuous preferences over $[0,\omega]$ (we make no monotonicity assumption: items can be desirable or not, preferences can be satiated etc..). Then *Equal Split* is the best Fair Guarantee: $Maxmin(u;n) = u(\frac{1}{n}\omega)$. To check this pick a hyperplane H supporting the upper contour of u at $\frac{1}{n}\omega$; then the *lower* contour of u at $\frac{1}{n}\omega$ contains one closed half-space cut by H, and every division of the manna as $\omega = \sum_1^n z_i$ includes at least one z_j in that half-space. This proves $Maxmin(u;n) \leq u(\frac{1}{n}\omega)$; the reverse inequality is clear. So the Equal Split Guarantee is the uncontroversial starting point of fairness, satisfied by the standard efficient and fair division rules like the competitive and egalitarian ones (surveyed in [37] and [27]). [2]

[2]Another example where *Maxmin* is a feasible guarantee: we divide manna and money

We generalize substantially both models above by allowing much more general utilities, and we identify a canonical Fair Guarantee this very large class of division problems.

As explained in Subsection 3.1, the microeconomic model is in fact a special case of the cake-cutting model two paragraphs above. Therefore we state our results in the latter model but use the more intuitive microeconomic examples for illustration. The manna Ω is measurable and endowed with a non atomic measure; utilities are continuous in that measure (small changes in the size of a share result in small utility changes) but otherwise arbitrary: neither additivity, nor sub- or super-modularity is assumed. We dub this a *non atomic division problem.* In our first main result utilities are not necessarily monotonic either: the manna can be a mixed bag with some desirable parts (money, tasty cake, valuable and resalable objects), some not (unpleasant tasks, financial liabilities, burnt parts of the cake that must still be eaten [32]), and agents may disagree over which parts are good or bad. Utilities can be single-peaked over some parts (teaching loads, volunteering time, shares of a risky project), single-dipped on others, and so on.

We show that the "dual" of the canonical upper bound $Maxmin(u;n)$ is a Fair Guarantee in our general model. This is $minMax(u;n)$, the utility of the best share for u in the worst possible n-partition of Ω:

$$minMax(u;n) = \min_{\Pi=(S_i)_{i=1}^n} \max_{1\leq i\leq n} u(S_i)$$

where as before the minimum bears on all n-partitions of Ω.

A simple microeconomic example where two agents Ann and Bob share 10 units of a divisible item shows that $Maxmin(u;n)$ is not a feasible Guarantee when preferences are not convex, and gives some intuition for our general result. Ann's preferences are single-peaked, while Bob's are single-dipped (see Figure 1):

$$u_A(x) = x(12-x) \;\; ; \;\; u_B(x) = x(x-6) \text{ for } 0 \leq x \leq 10$$

Compute

$Maxmin(u_A) = 35$ at partition $\Pi_1 = \{5,5\}$; $minMax(u_A) = 20$ at $\Pi_2 = \{0,10\}$

---

and utilities are quasi-linear (linear in money) and superadditive. We can use budget balanced monetary transfers to build a partition of Ω in which each share is worth $\frac{1}{n}u(\Omega)$: thus $Maxmin(u;n) \geq \frac{1}{n}u(\Omega)$. The reverse inequality follows from superadditivity.

See Remark 2 in Subsection 3.3 for another example in the interval model.

$$Maxmin(u_B) = 0 \text{ at } \Pi_2 \; ; \; minMax(u_B) = -5 \text{ at } \Pi_1$$

If Bob's share is worth at least $Maxmin(u_B)$ then either Ann gets the whole manna or at most 4 units: so her utility is at most 32 hence $(Maxmin(u_A), Maxmin(u_B))$ is not feasible. Note that, here and in general, both $(Maxmin(u_A), minMax(u_B))$ and $(minMax(u_A), Maxmin(u_B))$ are feasible: the former when Ann divides and Bob chooses, the latter when they exchange roles.

Our first main result, Theorem 1 in Section 4, says that in any non atomic problem, the mapping $u \to minMax(u; n)$ is a Fair Guarantee; in particular $minMax(u; n) \leq Maxmin(u; n)$ for all $u \in \mathcal{D}$ and n ((1)).[3] Moreover $minMax$ is implemented by a simple division rule described two paragraphs below.

There are two steps in the proof. The first is the critical "equi-partition" Lemma 1 in Subsection 3.2, stating that each agent can partition any subset of the manna in shares of equal value to him. Its proof requires advanced tools in algebraic geometry (see Appendix 7.1). The second step uses Kuhn's little known n-person version of Divide and Choose ([19]), denoted here D&C$_n$, to implement the $minMax$ Guarantee. This rule works as follows (details in Section 4):

Agent 1 cuts the manna in n shares presumed of equal value to her; each other agent reports which of these shares he finds acceptable. A simple matching algorithm gives one of "her" shares to agent 1, and some other shares to other agents who accept them (possibly none), making sure that the shares thus assigned are all unacceptable to the agents not served. Repeat with the remaining agents and manna.

As noted in the example above, in D&C$_2$ the Divider is guaranteed her $Maxmin$ utility, the Chooser his $minMax$ utility. The same applies to D&C$_n$ for any n: the *first* Divider is guarantees her $Maxmin$ utility, and *everyone else* his $minMax$ utility.

Our second main result, Theorem 2 in Subsection 5.2, focuses on non atomic problems where preferences are also *co-monotone*: *increasing* if enlarging a share cannot make it worse and we speak of a "good manna"; *decreasing* if the opposite holds and we have a "bad manna". The family of *Moving Knife* rules (MK$_n$) due to Dubins and Spanier ([18]) can be used in either domain. If the manna is good for everyone, they work as follows (details in Section 5):

[3] This is an equality if utilities are additive on Ω.

A knife cuts an increasing share of the cake; agents can stop the knife at any time; the *first* agent who does gets the share cut so far.
Repeat between the remaining agents and manna.

If the manna is unanimously bad, each agent can "drop" at any time and the *last* one to drop gets the share cut so far.

Under additive positive utilities each agent guarantees his Proportional Share by stopping the knife in each round exactly when it has cut a share worth $\frac{1}{n}$-th of the total. In the co-monotone version of our general non atomic model, it is easy to check that MK_n implements a Fair Guarantee between $minMax$ and $Maxmin$.

But a Moving Knife rule chooses a single arbitrary path for the knife, which tightly restricts the range of individual shares and partitions, hence can result in a very inefficient allocation. To alleviate this serious difficulty, we introduce a large family of rules in the same spirit that we call the *Bid & Choose* ($\mathrm{B\&C}_n$) rules. Each rule is defined by fixing a benchmark *additive measure* of the shares, diversely interpreted as their size, their market price, etc.. If the manna is good a bid b_i by agent i is the smallest measure of a share that i finds acceptable: the smallest bidder i^* chooses freely a share of measure at most b_{i^*}, then we repeat between the remaining agents and manna. For a bad manna the bid b_i is the largest size of a share that i finds acceptable, and the largest bidder i^* picks any share of size at least b_{i^*}.

Theorem 2 shows that each $\mathrm{B\&C}_n$ rule implements a Guarantee between the $minMax$ and $Maxmin$ level. A handful of microeconomic examples in Subsection 5.3 show that it improves substantially the $minMax$ Guarantee and is a legitimate alternative to the Equal Split Guarantee: the latter is optimal for agents with convex preferences, but for agents with "concave" preferences (convex lower contours) Equal Split is precisely the $minMax$ Guarantee, and the $\mathrm{B\&C}_n$ Guarantee is significantly better.

Throughout the paper we speak of implementation in the very simple sense adopted by most of the cake cutting literature (e. g., [13]), and formalized in the general collective decision context by Barbera and Dutta as implementation in "protective equilibrium" ([6]). A rule implements (guarantees) a certain utility level γ means: no matter what her preferences, each agent has a strategy that depends also upon the manna and the number of agents, but nothing else, such that whatever other agents do the utility of her share is no less than γ. Moreover the "guaranteeing strategy" is essentially unique.

2 Relevant literature

The two welfare levels $Maxmin$ and $minMax$ are key to our results. They are introduced by Budish ([15]) and Bouveret and Lemaitre ([11]) respectively in the *atomic* model where the manna is a set of indivisible items. If utilities are additive the basic inequality of our *non atomic* model is reversed:

$$Maxmin(u;n) \leq \frac{1}{n}u(\Omega) \leq minMax(u;n)$$

and $minMax(u;n)$ is obviously not a feasible Guarantee. It took a couple of years and many brain cells to check that also the $Maxmin$ lower bound is not feasible for three or more agents ([30]), though this happens in rare instances of the model ([21]).[4] Our paper is the first systematic discussion of these two bounds in the non atomic model of cake division.

Kuhn's 1967 n person generalisation of Divide and Choose ([19]) promptly implements the $minMax$ guarantee. Except for a recent discussion in [1] for additive utilities, it has not received much attention, a situation which our paper may help to correct. In particular, unlike the Diminishing Share ([34]), Moving Knife ([18]), and Bid and Choose rules, it is very well suited to divide *mixed manna,* i. e., containing subjectively good and bad parts, as is typically the case when we divide the assets and liabilities of a dissolving partnership. Agents applying the D&C$_n$ rule to a mixed manna are never asked to report which parts they view as good or bad: e. g. if we divide tasks, I may not want others to know which ones I am actually happy to perform. Introduced in [10] and [9] for the competitive fair division of commodities in the microeconomic model, the mixed manna model is discussed by [32] for a general cake, and by [4] for indivisible items.

The "equi-partition" Lemma (Subsection 3.2) is critical to the proof of Theorem 1, and proved by algebraic geometry techniques. Similarly, subtle variants of Sperner's Lemma are used in recent results to prove the existence of an Envy Free division under very general preferences, where which share I like best in a given partition can depend upon the partition itself, not just upon my own share: Stromquist's ([35]) and Woodall's ([38]) seminal insights are considerably strenghtened by the combined results in [36], [32], [24] and [3]. However all these results assume that, either no agent prefers the empty

[4] If the manna is atomic and utilities are not necessarily additive, it is easy to construct examples showing that all six orderings of $Maxmin$, $minMax$, and $\frac{1}{n}u(\Omega)$ are possible.

share to a non zero share, or all agents always prefer the empty share to any non zero share: this rules out the mixed manna case.

The concept of *unanimity utility* (the common efficient utility level in the economy where everyone has the same preferences) leads to the Equal Split Guarantee when we divide private goods and preferences are convex (see the proof of inequality (1)).When applied to fair division problems involving production, it defines some compelling Fair Guarantees as well as some minimal *upper bounds* on individual welfare: [26], [25].

Privacy preservation is a growing concern in a world of ever expanding information flows. The iconic Divide and Choose rule between two agents stands out for its informational parsimony: Divider only reports, whether this is true or not, that she is indifferent between the two shares of the cake she just cut, then Chooser reveals only the binary comparison of these shares. A related advantage is that each report requires a modest cognitive effort, neither Divider nor Chooser needs to form complete preference relations over all shares of the cake. These properties are preserved by the n person version D&C$_n$ of the rule.

Taking this concern to heart, the large cake cutting literature following Steinhaus' seminal paper evaluates the informational complexity of various mechanisms by the number of "cuts" and "queries" they involve: see [13] or [31], and more recently [16] and [17]. It also goes beyond the Proportional Guarantee, and looks for cuts and queries mechanisms reaching an Envy Free division of the cake. The algorithms in Brams and Taylor ([12]), and more recently Aziz and McKenzie ([5]), do exactly this when utilities are additive and non atomic; but because they involve an astronomical number of cuts and queries they are of no practical interest and squarely contradict informational parsimony. See ([14], [20]) for some fine tuning of these general facts.

3 Non atomic fair division

3.1 Basic definitions

The manna Ω is a bounded measurable set in an euclidian space, endowed with the Lebesgue measure $|\cdot|$, and such that $|\Omega| > 0$. A share S is a possibly empty measurable subset of Ω, and $\mathcal{B}$ is the set of all shares. A n-partition of $\otimes$ is a n-tuple of shares $\Pi = (S_i)_{i=1}^n$ such that $\cup_{i=1}^n S_i = \Omega$ and $|S_i \cap S_j| = 0$

for all $i \neq j$; and $\mathcal{P}_n(\Omega)$ is the set of all partitions of $\otimes$. We define similarly an *n-partition of* S for any share $S \in \mathcal{B}$, and write their set as $\mathcal{P}_n(S)$.

If $S \otimes T = (S \cup T) \diagdown (S \cap T)$ is the symmetric difference of shares, recall that $\delta(S,T) = |S \otimes T|$ is a pseudo-metric on $\mathcal{B}$ (a metric except that $\delta(S,T) = 0$ iff S and T differ by a set of measure zero).

A utility function u is a mapping from $\mathcal{B}$ into $\mathbb{R}$ such that $u(\varnothing) = 0$ and u is continuous for the pseudo-metric δ and bounded. So u does not distinguish between two shares at pseudo-distance zero (equal up to a set of measure zero): for instance $u(S) = 0$ if $|S| = 0$. Also if the sequence $|S^t|$ converges to zero in t, so does $u(S^t)$. We write $\mathcal{D}(\Omega)$ for this domain of utility functions.

So a non atomic division problem consists of $(\Omega, \mathcal{B}, (u_i)_{i=1}^n \in \mathcal{D}(\Omega)^n)$.

Several subdomains of $\mathcal{D}(\Omega)$ play a role below:

- additive utilities: $u \in \mathcal{A}dd(\Omega)$ iff $u(S) = \int_S f(x)dx$ for all S, where f is bounded and measurable in Ω;

- monotone increasing: $u \in \mathcal{M}^+(\Omega)$ iff $S \subset T \Longrightarrow u(S) \leq u(T)$ for all S, T;

- monotone decreasing: $u \in \mathcal{M}^-(\Omega)$ iff $S \subset T \Longrightarrow u(S) \geq u(T)$ for all S, T;

- separable: $u \in \mathcal{S}(\Omega)$ iff there is a finite set A, a partition $(C_a)_{a \in A} \in \mathcal{P}_{|A|}(\otimes)$ of Ω, and a continuous function v from $\mathbb{R}_+^A$ into $\mathbb{R}$, such that $u(S) = v((|S \cap C_a|)_{a \in A})$ for all $S \in \mathcal{B}$.

The separable domain $\mathcal{S}(\Omega)$ captures the standard microeconomic fair division model: A is a set of divisible commodities, the manna is the bundle $\omega \in \mathbb{R}_+^A$ such that $\omega_a = |C_a|$ for all a, a share S_i gives to agent i the amount $z_{ia} = |S_i \cap C_a|$ of commodity a, and the partition $\Pi = (S_i)_{i=1}^n$ corresponds to the division of the manna as $\omega = \sum_1^n z_i$.

In the general non atomic division problem, the set of shares $\mathcal{B}$ is not compact for the pseudo-metric δ. It follows that when we maximize or minimize utilities over shares, or look for a partition achieving a benchmark utility $minMax$ or $Maxmin$, we cannot claim the existence of an exact solution to the program: the $minMax$ is not a true minimum, only an infimum, and $Maxmin$ is only a supremum, not a true maximum. As this will cause no confusion, we stick to the min and Max notation throughout.

However in the microeconomic model, the set of shares and of partitions are both compact so for this important set of problems (where all our examples live) the *min* and *Max* notation are strictly justified.

One can also specialise the general model by imposing constraints on the set of feasible shares. The most important instance is the familiar *interval model*, where the manna is $\Omega = [0, 1]$ and a share must be an interval, so an n-partition is made of n adjacent intervals. Other instances assume Ω is a polytope, and shares are polytopes of a certain type: e.g. triangles or tetrahedrons ([33]). And sometimes shares must be connected subsets of Ω ([7], [2]).

The Divide and Choose$_n$ rules, as well as our Bid and Choose$_n$ rules, do not work in these models[5], so our Theorems 1 and 2 do not apply. But the interval model is still useful here in a technical sense: the proof of the critical Lemma 1 in Appendix 7.1 starts by projecting the general problem onto an interval model and proving existence of an equipartition there.

3.2 Equi-partitions

Definition 1 *An n-equipartition of the share $T \in \mathcal{B}$ for utility $u \in \mathcal{D}(T)$ is a partition $\Pi^e = (S_i)_{i=1}^n \in \mathcal{P}_n(T)$ such that $u(S_i) = u(S_j)$ for all $i, j \in \{1, \cdots, n\}$; we write $u(\Pi^e)$ for this common value, and $\mathcal{EP}_n(T; u)$ for the set of these n-equipartitions.*

It is clear that $\mathcal{EP}_n(S; u)$ is non empty if u is additive: if $\mathcal{B}[\mathcal{S}]$ is the subset of shares included in S, Lyapunov Theorem implies that the range $u(\mathcal{B}[\mathcal{S}])$ is convex, so it contains $\frac{1}{n}u(S)$; then we replace n by $n-1$ and repeat the argument on the remaining share.

The same is true if u is monotone ($u \in \mathcal{M}^{\pm}(\Omega)$), and the proof, outlined in Remark 1 below, is still fairly simple. But the proof of the next statement is much harder.

Lemma 1
Fix a share $S \in \mathcal{B}$ and a utility $u \in \mathcal{D}(\Omega)$. The set $\mathcal{EP}_n(S; u)$ of n-equipartitions of S at u is non empty.

Remark 1 It is easy to prove $\mathcal{EP}_n(S; u) \neq \varnothing$ if we assume that the sign of u is constant: all shares are weakly preferred to the empty share, or all

[5] For instance in the interval model, the first divider can find an equipartition made of adjacent intervals (by our Lermma 1), but the next agent called to divide typically cannot do so pick from disconnected intervals.

are weakly worse. Use first a Moving Knife as in Appendix 7.1 to project the model onto an interval model: in the latter model, a partition is identified with a point in the simplex of dimension $n-1$. We apply the Knaster–Kuratowski–Mazurkiewicz Lemma to the closed sets Q_i of partitions of the interval where the i-th interval gives the lowest utility: each Q_i contains the i-th face of the simplex and their union covers it entirely: thus their intersection is non empty. One can also invoke the stronger results in [35] and [36] showing the existence of an Envy Free partition under this assumption. But recall that a key feature in the division of a mixed manna is that the sign of u is ***not*** *constant across shares.*

3.3 Two utility benchmarks

Definition 2 *Fix n, the manna $(\Omega, \mathcal{B})$ and $u \in \mathcal{D}(\Omega)$:*

$$minMax(u;n) = \min_{\Pi \in \mathcal{P}_n(\Omega)} \max_{1 \leq i \leq n} u(S_i) \ ; \ Maxmin(u;n) = \max_{\Pi \in \mathcal{P}_n(\Omega)} \min_{1 \leq i \leq n} u(S_i) \tag{2}$$

Recall that $minMax$ is the utility agent u can achieve by having first pick in the worst possible n-partition of Ω, and $Maxmin$ by having last pick in the best possible n-partition of Ω.

Proposition 1
i) If $u \in \mathcal{A}dd(\Omega)$ then $minMax(u;n) = Maxmin(u;n) = \frac{1}{n}u(\Omega)$
ii) If $u \in \mathcal{M}^{\pm}(\Omega)$

$$minMax(u;n) = \min_{\Pi^e \in \mathcal{EP}_n(\Omega;u)} u(\Pi^e) \ ; \ Maxmin(u;n) = \max_{\Pi^e \in \mathcal{EP}_n(\Omega;u)} u(\Pi^e) \tag{3}$$

iii) If $u \in \mathcal{D}(\Omega)$

$$minMax(u;n) \leq \min_{\Pi^e \in \mathcal{EP}_n(\Omega;u)} u(\Pi^e) \leq \max_{\Pi^e \in \mathcal{EP}_n(\Omega;u)} u(\Pi^e) \leq Maxmin(u;n) \tag{4}$$

Proof
Statement iii) If Π^e is an n-equipartition, $u(\Pi^e)$ is the utility of its best share, hence $minMax(u;n) \leq u(\Pi^e)$; proving the other inequality in (4) is just as easy.
Statement i) By additivity of u, for any n-partition Π we have $\max_i u(P_i) \geq \frac{1}{n}u(\Omega)$ implying $minMax(u;n) \geq \frac{1}{n}u(\Omega)$; we check symmetrically $\frac{1}{n}u(\Omega) \geq$

$Maxmin(u;n)$, and the conclusion follows by comparing these inequalities to those in (4).

Statement ii) We assume $u \in \mathcal{M}^+(\Omega)$ without loss of generality. The continuity and monotonicity of u imply: if S, T are two disjoints shares such that $u(S) > u(T)$, we can trim part of S and add it to T to get two disjoint shares with equal utility in between $u(S)$ and $u(T)$. Expanding this argument, if $S_1, \cdots, S_k$ and T are disjoint shares such that

$$u(S_1) = u(S_2) = \cdots = u(S_k) > u(T)$$

we can simultaneously trim $S_1, \cdots, S_k$ keeping them of equal utility and add the trimming to T, so that the resulting $k+1$ shares are all equally good and their common utility is between the two utilities above. Iterating this process, we see that if $\Pi = (S_i)_{i=1}^n \in \mathcal{P}_n(\Omega)$ is such that $\max_{1\leq i\leq n} u(S_i) > \min_{1\leq i\leq n} u(S_j)$, we can construct an equi-partition $\Pi^e \in \mathcal{EP}_n(\Omega; u)$ such that

$$\max_{1\leq i\leq n} u(S_i) > u(\Pi^e) > \min_{1\leq j\leq n} u(S_j)$$

Now fix $\varepsilon > 0$, arbitrarily small, pick $\Pi = (S_i)_{i=1}^n \in \mathcal{P}_n(\Omega)$ such that $\min_{1\leq j\leq n} u(S_j) \geq Maxmin(u;n) - \varepsilon$, and assume that Π is not an equi-partition. By the argument above we can find $\Pi^e \in \mathcal{EP}_n(\Omega; u)$ such that $u(\Pi^e) > \min_{1\leq j\leq n} u(S_j)$, therefore Π^e too is an ε-approximation of $Maxmin(u;n)$, and the right-hand inequality in (3) follows. The proof of the left-hand inequality is similar. ■

In the general domain $\mathcal{D}(\Omega)$, the partitions achieving the $Maxmin$ and $minMax$ utilities are not necessarily equi-partitions. In the microeconomic example of Section 1 we divide $\omega = 10$ units of a single commodity between two agents. Ann has single-peaked preferences and her $minMax$ is achieved by the all-or-nothing partition $\{\varnothing, \Omega\}$; Bob has single-dipped preferences and the same partition delivers his $Maxmin$; and $\{\varnothing, \Omega\}$ is not an equipartition for either utility.

Remark 2: In the interval model with a monotone utility u, it is easy to check that any two n-equipartitions have the same utility and in turn this implies $minMax(u;n) = Maxmin(u;n)$ and this is the best Fair Guarantee. The numerical example above can be viewed as an instance of the interval model where the two agents are indifferent between $[0,x]$ and $[1-x,1]$ for all x: so only the inequality (4) holds true in the general (non monotone) interval model.

3.4 Fair Guarantees

Definition 3 *Fix the manna* $(\Omega, \mathcal{B})$ *and a subdomain* $\mathcal{D}^*$, $\mathcal{D}^* \subseteq \mathcal{D}(\Omega)$. *An Fair Guarantee in* $\mathcal{D}^*$ *is a mapping* $\Gamma : u \to \Gamma(u; n)$ *such that for any profile* $(u_i)_{i=1}^n \in (\mathcal{D}^*)^n$ *there exists* $\Pi = (S_i)_{i=1}^n \in \mathcal{P}_n(\Omega)$ *such that* $u_i(S_i) \geq \Gamma(u_i; n)$ *for all* i.

By looking in Section 1 at the u-unanimity profile we observed that $Maxmin$ is an upper bound of *any* Fair Guarantee (inequality (1)). We also mentioned two subdomains where $Maxmin$ itself is a (hence the optimal) Fair Guarantee: the additive domain $\mathcal{A}dd(\Omega)$ and the subdomain of the separable one $\mathcal{S}(\Omega)$ where preferences are also convex. Finally we used the Ann and Bob microeconomic example with a single commodity to show that $Maxmin$ is not a Fair Guarantee in $\mathcal{D}(\Omega)$, even for $n = 2$ and a one dimensional manna.

Before proving in the next Section that $minMax(\cdot; n)$ is a Fair Guarantee in the whole domain $\mathcal{D}(\Omega)$ we construct a microeconomic example with two divisible items and two agents u_1 and u_2 where

$$\begin{gathered} minMax(u_i; 2) = 0 < 1 = Maxmin(u_i; 2) \text{ for } i = 1, 2 \\ \text{and } (minMax(u_1), minMax(u_2)) \text{ is weakly Pareto optimal} \end{gathered}$$

Thus for any Fair Guarantee Γ at least one of $\Gamma(u_1; 2) = 0$ and $\Gamma(u_2; 2) = 0$ must hold. In words, for some problems, no Fair Guarantee can reduce the gap from $minMax$ to $Maxmin$ for both agents.[6]

We divide one unit of each item, $\omega = (1, 1)$ and write shares as $z = (x, y)$. Both utilities are symmetric in x, y: $u_i(x, y) = u_i(y, x)$ so we it is enough to define them for $x \leq y$:

$$\begin{aligned} u_1(z) &= \quad 0 & &\text{if } x \leq \tfrac{1}{2} \leq y \\ u_1(z) &= 1 - 2y & &\text{if } x \leq y \leq \tfrac{1}{2} \\ u_1(z) &= 2x - 1 & &\text{if } \tfrac{1}{2} \leq x \leq y \end{aligned}$$

$$\begin{aligned} u_2(z) &= \quad 0 & &\text{if } x \leq y \leq \tfrac{1}{2} \text{ or } \tfrac{1}{2} \leq x \leq y \\ u_2(z) &= 2y - 1 & &\text{if } \tfrac{1}{2} \leq y \leq 1 - x \\ u_2(z) &= 1 - 2x & &\text{if } \tfrac{1}{2} \leq 1 - x \leq y \end{aligned}$$

The range of both utilities is $[0, 1]$. Agent 1's utility $u_1(z_1)$ is null in the NW and SE quadrant of the box $[0, 1]^2$ with center at $(\frac{1}{2}, \frac{1}{2})$; it is strictly positive

[6] Of course Divide and Choose implements the utility profile $(minMax(u_i; 2), Maxmin(u_j; 2))$ so this gap can be closed for one agent.

in the SW and NE quadrants except on the lines $x = \frac{1}{2}$ and $y = \frac{1}{2}$. Agent 2's utility $u_2(z_2)$ is symetrically null in the SW and NE quadrants, and strictly positive in the NW and SE quadrants except on the same two lines. Therefore for any division $(1,1) = z_1 + z_2$ of the manna we have $u_1(z_1) \cdot u_2(z_2) = 0$: there is no feasible division s. t. $u_i(z_i) > 0$ for $i = 1, 2$.

The partition $\{(0,0),(1,1)\}$ achieves $Maxmin(u_1) = 1$ and $minMax(u_2) = 0$; the partition $\{(0,1),(1,0)\}$ achieves $Maxmin(u_2) = 1$ and $minMax(u_1) = 0$.

4 The Divide & Choose$_n$ rule

We start by a combinatorial observation. Let G be a bilateral graph between the sets M of agents and R of shares: $(m, r) \in G$ means agent m *likes* share r. We say that the subset $\widetilde{M}$ of agents are *properly matched* to the subset $\widetilde{R}$ of shares if $|\widetilde{M}| = |\widetilde{R}|$, agents in $\widetilde{M}$ are each matched (one-to-one) to a share they like in $\widetilde{R}$, and no one outside $\widetilde{M}$ likes any share in $\widetilde{R}$.

Lemma 2. *Assume $|M| = |R|$, each agent in M likes at least one object in R and some agent i^* likes all objects in R. Then there is a (non empty) largest set M^* of properly matchable agents containing i^*: if $\widetilde{M}$ is properly matched to $\widetilde{R}$, then $\widetilde{M} \subseteq M^*$.*

Proof. This is a simple consequence of the Gallai-Edmonds decomposition of a bipartite graph: see e.g. [22] Chap 3 (or Lemma 1 in [8]). If M can be matched with R this is a proper match and the statement holds true. If M and R cannot be matched, then we can uniquely partition M as (M^+, M^*) and R as (R^+, R^*) such that:

1. $|M^+| > |R^+|$, the agents in M^+ do not like any object in R^*, and they compete for the over-demanded objects in R^+: every subset of R^+ is liked by a strictly larger subset of agents in M^+;
2. $|M^*| < |R^*|$ and the agents in M^* can be matched with some subset of R^*.

By the general Gallai-Edmonds result, M^+ and R^* are non empty. Here M^* is non empty as well because it contains the special agent i^*. Every match of M^* to a subset of R^* is proper. Finally suppose $\widetilde{M}$ is properly matched to $\widetilde{R}$ and $\widehat{M} = \widetilde{M} \cap M^+$ is non empty. Then $\widehat{M}$ is matched to some subset $\widehat{R}$ of R^+ but $\widehat{R}$ is liked by more agents in M^+ than there are in $\widehat{M}$, therefore the match is not proper: contradiction. So $\widetilde{M}$ does not intersect

M^+ as was to be proved.■

Definition 4: *the D&C_n rule.*
Fix the manna $(\Omega, \mathcal{B})$ *and the ordered set of agents* $N = \{1, \cdots, n\}$*, each with a utility in* $\mathcal{D}(\Omega)$*.*
Step 1. Agent 1 *proposes a partition* $\Pi^1 \in \mathcal{P}_n(\Omega)$*; all other agents report which shares in* Π^1 *they like (at least one). In the resulting bipartite graph between* N *and the shares in* Π^1*, where agent* 1 *likes all the shares, we use Lemma 2 to match properly the largest possible set of agents* N^1 *(it contains agent* 1*) with some set of shares* R*; if* $N^1 = N$ *we are done, otherwise* we *go to*
Step 2. Repeat with the remaining manna Ω^2 *and agents in* $N \diagdown N^1$*. Ask the first agent in the exogenous ordering to propose a partition* $\Pi^2 \in \mathcal{P}_{n-|N_1|}(\Omega^2)$*, while others report which of these new shares they like. And so on.*

At least one agent, the Divider, is served in each step, thus the algorithm just described takes at most $n-1$ steps.

There is some flexibility in the Definition of the rule: although the set of agents matched in each step is unambiguous, we have typically several choices for the set R of shares to assign in each step, and multiple ways to assign these to the agents. We match as many agents as possible so as to minimize the number of queries, hence of information disclosure.

Our first main result is that $minMax$ is a Fair Guarantee, implemented by the D&C_n rule in the full domain $\mathcal{D}(\Omega)$.

Theorem 1

Fix the manna $(\Omega, \mathcal{B})$ *and* n*.*
i*) In the D&*C_n *rule, an agent with utility* $u \in \mathcal{D}(\Omega)$ *guarantees the* $minMax(u; n)$ *utility level by 1) when called to divide, proposing an equi-partition* $\Pi^e \in \mathcal{EP}_m(S)$ *of the remaining share* S *of manna among the* m *remaining agents, and 2) when reporting shares he likes, accepting* ***all*** *shares, and only those, not worse than* $minMax(u; n)$ *(the* $minMax$ *level in the* ***initial*** *problem).*
ii*) Moreover the first Divider (and no one else) can guarantee her* $Maxmin$ *utility. Other agents cannot guarantee more than their* $minMax$ *utility.*

Proof. *Statement* i). Consider agent u using the strategy in the statement. At a step where he must report which shares he likes among those offered at that step, he can for sure find one worth at least $minMax(u; n)$: all shares previously assigned are worth to him strictly less than $minMax(u; n)$, and together with the freshly cut shares they form a partition in $\mathcal{P}_n(\Omega)$; in any partition at least one share is worth $minMax(u; n)$ or more.

At a step where our agent is called to cut, he proposes to the remaining agents an m-equi-partition $\Pi^e \in \mathcal{EP}_m(S)$ of the remaining manna S. To check the inequality $u(\Pi^e) \geq minMax(u;n)$ note that Π^e together with the previously assigned shares is a partition of Ω in which the old shares are worth strictly less than $minMax(u;n)$.[7]

Statement ii). This is clear for the first Divider. Fix now an agent i with utility u and check that if he is not the first Divider, for certain moves of the other agents, agent u gets exactly his $minMax$ utility. Pick a partition $\Pi \in \mathcal{P}_n(\Omega)$ achieving $minMax(u;n)$ (the existence assumption is without loss). Suppose that the first Divider, who is not agent i, offers Π, and all agents other than i (but including the Divider) find all shares acceptable: then a full match is feasible (i must accept at least one share) so his share cannot be worth more than $minMax(u;n)$. ■

5 Bid and Choose and Moving Knives for good or bad manna

We now assume that the manna is unanimously good, $u \in \mathcal{M}^+(\Omega)$, or unanimously bad, $u \in \mathcal{M}^-(\Omega)$. Because $u(\varnothing) = 0$, for all S we have $u(S) \geq 0$ in the former case and $u(S) \leq 0$ in the latter. Recall that in these two domains, the $minMax$ (resp. $Maxmin$) utility is the smallest (resp. largest) equi-partition utility: property (3) in Proposition 1.

We check first that the profile of $Maxmin$ utility levels still may not be feasible, even in the simple microeconomic model (corresponding to the separable domain defined in Subsection 3.1). The manna $\omega = (1,1)$, one unit each of two divisible goods, is shared by two agents and a share is written $z = (x,y)$. The first one has Leontief preferences $u_1(z) = \min\{x,y\}$ so his worst case partition is $\Pi = \{(1,0),(0,1)\}$ and his best one is the equal split partition $\Pi' = \{\frac{1}{2}\omega, \frac{1}{2}\omega\}$: $minMax(u_1;2) = 0 < \frac{1}{2} = Maxmin(u_1;2)$. Agent 2 has anti-Leontief preferences: $u_2(z) = \max\{x,y\}$. For her the equal split partition Π' is the worst and the best one is Π: $minMax(u_2;2) = \frac{1}{2} < 1 = Maxmin(u_2;2)$. Thus the profile of $Maxmin$ utilities $(\frac{1}{2},1)$ is not feasible, and D&C$_2$ guarantees only the $minMax$ utilities $(0,\frac{1}{2})$.

[7] After Step 1 an agent can secure his $Maxmin$ utility for the smaller manna S among m agents, but this may be below the $Maxmin$ utility in the initial problem.

We show that the $minMax$ guarantee is always improved, at least weakly, by the large family of Bid and Choose (B&C$_n$) rules, inspired by the familiar Moving Knives (MK$_n$) rules ([18]).

5.1 MK$_n^\kappa$ and B&C$_n^\theta$ rules

A *moving knife* through the manna $(\Omega, \mathcal{B}, |\cdot|)$ is a path $\kappa : [0,1] \ni t \to K(t) \in \mathcal{B}$ from $K(0) = \varnothing$ to $K(1) = \Omega$, continuous for the pseudo-metric δ on $\mathcal{B}$ and strictly inclusion increasing:

$$0 \leq t < t' \leq 1 \Longrightarrow K(t) \subset K(t') \text{ and } |K(t') \diagdown K(t)| > 0$$

The moving knife κ arranges shares of increasing value to all participants along the specific path of the knife. An example is $K(t) = B(t) \cap \Omega$, where $t \to B(t)$ is a path of balls with a fixed center and radius growing from 0 to 1, so that $B(1)$ contains Ω. Moving knifes can take many other shapes, for instance hyperplanes.

Our Bid and Choose rules offer more choices to the agents, with the help of a benchmark *measure* θ of the shares, chosen by the rule designer: θ is a positive σ-additive measure on $(\Omega, \mathcal{B})$, normalised to $\theta(\Omega) = 1$. It is absolutely continuous w.r.t. the Lebesgue measure $|\cdot|$ and vice versa: the density of θ w.r.t. $|\cdot|$ is strictly positive. In particular θ is strictly inclusion increasing:

$$\forall S, T \in \mathcal{B} : S \subset T \text{ and } |T \diagdown S| > 0 \Rightarrow \theta(S) < \theta(T)$$

In applications θ can evaluate for instance the market value, physical size, or weight of a share.

Fixing a moving knife κ and a measure θ, we define in parallel the Moving Knife (MK$_n^\kappa$) and the Bid and Choose (B&C$_n^\theta$) rules. In both cases a clock t runs from $t = 0$ to $t = 1$.

Definition 5 *the MK$_n^\kappa$ and B&C$_n^\theta$ rules with increasing utilities*
Step 1. The first agent i_1 to stop the clock, at t^1, gets the share $K(t^1)$ in MK$_n^\kappa$, or in B&C$_n^\theta$ chooses any share in Ω s.t. $\theta(S) = t^1$, say S_{i_1}, and leaves;
Step k: Whoever stops the clock first at t^k gets the share $K(t^k) \diagdown K(t^{k-1})$ in MK$_n^\kappa$, or in B&C$_n^\theta$ chooses any share in $\Omega \diagdown \cup_1^{k-1} S_{i_\ell}$ s.t. $\theta(S) = t^k - t^{k-1}$, say S_{i_k}, and leaves;
In Step $n-1$ the single remaining agent who did not stop the clock takes the remaining share $\Omega \diagdown K(t^{n-1})$ or $\Omega \diagdown \cup_1^{n-1} S_{i_\ell}$.

Definition 5* *with decreasing utilities*
In each step all agents must choose a time to "drop", and the last agent i_1 who drops, at t^1, gets $K(t^1)$ in MK_n^κ, or in $B\&C_n^\theta$ chooses S_{i_1} s.t. $\theta(S_{i_1}) = t^1$. The other steps are similarly adjusted.

Breaking ties between agents stopping the clock (or dropping) at the same time is the only indeterminacy in these rules, much less severe than in D&C$_n$, where we serve at each step an unambiguous set of agents, but there are typically several ways to match them properly.

Up to tie-breaking, B&C$_n^\theta$ and MK$_n^\kappa$ are anonymous (do not discriminates between agents) but not neutral (do discriminate between shares), while D&C$_n$ is neutral but not anonymous.

The range of the B&C$_n^\theta$ rule is the entire set $\mathcal{P}_m(\Omega)$ of partitions of the manna: this stands in sharp contrast with MK$_n^\kappa$ in which the set of feasible shares for an agent (resp. feasible partitions) is of dimension 2 (resp. $n-1$) . To check the former claim we fix $\Pi = (S_i)_{i=1}^n$ and assume first $|S_i| > 0$ for all i. Consider n agents deciding (cooperatively) to achieve Π. By the strict monotonicity of θ the sequence $t^i = \theta(\cup_{j=1}^i S_j)$ increases strictly therefore they can stop the clock (or drop) at these successive times and choose the corresponding shares in Π. If there are shares of measure zero they can all be distributed at time 0.

On the other hand in B&C$_n^\theta$ all but one agent must pick a share under constraints, thus revealing more information than in MK$_n^\kappa$. Loosely speaking, B&C$_n^\theta$ is informationally comparable to D&C$_n$.

Remark 3. We can also implement the same Guarantees described in the next Subsection by alternative static versions of MK_n^κ and *$B\&C_n^\theta$ where agents bid all at once for potential stopping times; we do not discuss these rules for the sake of brevity.*

5.2 B&C$^\theta$ and MK$^\kappa$ Guarantees

We fix an increasing utility $u \in \mathcal{M}^+(\Omega)$. The results are identical, and identically phrased, for a bad manna $u \in \mathcal{M}^-(\Omega)$. See also Remark 4 at the end of this Subsection.

Define the triangle $\mathcal{T} = \{(t^1, t^2) | 0 \leq t^1 \leq t^2 \leq 1\}$ in $\mathbb{R}_+^2$ and the set $\Upsilon(n)$ of increasing sequences $\tau = (t^k)_{0 \leq k \leq n}$ in $[0, 1]$ s.t.

$$t^0 = 0 \leq t^1 \leq \cdots \leq t^{n-1} \leq 1 = t^n$$

For a moving knife κ, utilities of the shares in MK_n^κ are described by the function u^κ on $\mathcal{T}$:

$$u^\kappa(t^1, t^2) = u(K(t^2) \diagdown K(t^1)) \text{ for all } (t^1, t^2) \in \mathcal{T}$$

For a measure θ, the corresponding definition in $\text{B\&C}^\theta$ is the *indirect utility* u^θ:

$$u^\theta(t^1, t^2) = \min_{T:\theta(T)=t^1} \max_{S:S\cap T=\varnothing;\theta(S)=t^2-t^1} u(S) \text{ for all } (t^1, t^2) \in \mathcal{T} \tag{5}$$

Both u^κ and u^θ decrease (weakly) in t^1 and increase (weakly) in t^2.

We show below that the Guarantees Γ^k and Γ^θ implemented by MK_n^κ and $\text{B\&C}_n^\theta$ respectively are computed as follows:

$$\Gamma^\alpha(u; n) = \max_{\tau\in\Upsilon(n)} \min_{0\leq k\leq n-1} u^\alpha(t^k; t^{k+1}) \text{ where } \alpha \text{ is } \kappa \text{ or } \theta \tag{6}$$

For instance in MK_2^κ with two agents we write τ_κ for the (not necessarily unique) position of the knife making our agent indifferent between the share $K(\tau_\kappa)$ and its complement:

$$\Gamma^\kappa(u; 2) = \max_{0\leq t^1\leq 1} \min\{u(K(t^1)), u(\Omega \diagdown K(t^1)) = u(K(\tau_\kappa)) = u(\Omega \diagdown K(\tau_\kappa)$$

In $\text{B\&C}_2^\theta$ the bid τ_θ makes the best share of size τ_θ as good as the worst share of size $1 - \tau_\theta$:

$$\Gamma^\theta(u; 2) = \max_{0\leq t^1\leq 1} \min\{\max_{\theta(S)=t^1} u(S), \min_{\theta(S)=t^1} u(\Omega \diagdown S)\} = \max_{\theta(S)=\tau_\theta} u(S) = \min_{\theta(S)=\tau_\theta} u(\Omega \diagdown S) \tag{7}$$

Lemma 4

i) The utility u^κ and the indirect utility u^θ are continuous. Both the minimum and maximum in (5) are achieved.

ii) The maximum of problem (6) (for both rules) is achieved at some $\tau \in \Upsilon(n)$ where the sequence t^k increases in k, all the $u^\alpha(t^k; t^{k+1})$ are equal, and this common utility is the optimal value of (6).

Proof in Appendix 7.2.

Theorem 2

Fix the manna $(\Omega, \mathcal{B})$, the number of agents n, and a utility $u \in \mathcal{M}^+(\Omega)$.

i) With the MK_n^κ rule, an agent guarantees the utility $\Gamma^\kappa(u; n)$ by committing to stop the knife at t_κ^k if exactly $k - 1$ other agents have been served before;

ii) With the B&C$_n^\theta$ rule, she guarantees $\Gamma^\theta(u;n)$ *by stopping the clock at* t_θ^k *if exactly* $k-1$ *other agents have been served before; and choosing then the best available share of size* $t^k - t^{k-1}$.
iii) $minMax(u;n) \leq \Gamma^\alpha(u;n) \leq Maxmin(u;n)$ where α is κ or θ.

Proof.
Statement i) and iii) for MK_n^κ. Recall the equi-partition $\Pi = (K(t_\kappa^k)\diagdown K(t_\kappa^{k-1}))_1^n$ has $u(\Pi) = \Gamma^\kappa(u;n)$. Thus (3) in Proposition 1 implies the inequalities *iii*). Next if the knife has been stopped $k-1$ times before our agent is served, the last stop occured at or before t_κ^{k-1} therefore if she does stop the knife at t_κ^k (and wins the possible tie break) her share is at least $K(t_\kappa^k)\diagdown K(t_\kappa^{k-1})$. If she never gets to stop the knife, the last stop is at or before t_κ^{n-1} and she gets at least $\Omega\diagdown K(t_\kappa^{n-1})$.

Statement ii). If she is the first to stop the clock (perhaps also winning the tie break) at step k, in step $k-1$ the clock stopped at $t^{k-1} \leq t_\theta^{k-1}$ and the share T already distributed at that time has $\theta(T) = t^{k-1}$: therefore she can choose a share with utility $u^\theta(t^{k-1};t_\theta^k) \geq u^\theta(t_\theta^{k-1};t_\theta^k) = \Gamma^\theta(u;n)$. If she is the last to be served, having never stopped the clock (or lost some tie breaks) the share assigned to all other agents has $\theta(T) = t^{n-1} \leq t_\theta^{n-1}$ therefore her share is worth $u^\theta(t^{n-1};1) \geq u^\theta(t_\theta^{n-1};1) = \Gamma^\theta(u;n)$.

Statement iii) for B&C$_n^\theta$.

Right hand inequality. It is enough to construct a partition $\Pi = (S_k)_1^n$ in which the utility of every share $S_k, 0 \leq k \leq n-1$ is at least $u^\theta(t_\theta^{k-1}, t_\theta^k)$, implying $\min_k u(S_k) \geq \Gamma^\theta(u;n)$. We proceed by induction on the steps of B&C$_n^\theta$. First S_1 maximizes $u(S)$ s.t. $\theta(S) = t_\theta^1$ so $u(S_1) = u^\theta(0;t_\theta^1) = \Gamma^\theta(u;n)$ and $\theta(S_1) = t_\theta^1$. Assume the sets S_ℓ are constructed for $1 \leq \ell \leq k$, mutually disjoint, s.t. $\theta(S_\ell) = t_\theta^\ell - t_\theta^{\ell-1}$ and $u(S_\ell) \geq u^\theta(t_\theta^\ell, t_\theta^{\ell-1})$: then the set $T = \cup_1^k S_\ell$ is of size t_θ^k and we pick S_{k+1} maximizing $u(S)$ s.t. $S \cap T = \varnothing$ and $\theta(S) = t_\theta^{k+1} - t_\theta^k$. By definition (5) we have $u(S_k) \geq u^\theta(t_\theta^k;t_\theta^{k+1})$ and the induction proceeds. Note that in fact $\min_k u(S_k) = \Gamma^\theta(u;n)$.

Left hand inequality. We need now construct a partition $\Pi = (R_k)_1^n$ s. t. $u(R_k) \leq u^\theta(t_\theta^{k-1};t_\theta^k)$ for $1 \leq k \leq n$. We do this by a decreasing induction in n. In (the first) step n of the induction we define the 2-partition $\Pi^n = (T_{n-1}, R_n)$ of Ω where T_{n-1} is any solution of the program $\min_{T:\theta(T)=t_\theta^{n-1}} u(\Omega\diagdown T)$, and $R_n = \Omega\diagdown T_{n-1}$. Thus $u(R_n) = u^\theta(t_\theta^{n-1};1)$ and $\theta(T_{n-1}) = t_\theta^{n-1}$.

Assume that in step k we constructed the $(n-k+2)$-partition $\Pi^k = (T_{k-1}, R_k, R_{k+1}, \cdots, R_n)$ s.t. $\theta(T_{k-1}) = t_\theta^{k-1}$ and $u(R_\ell) \leq u^\theta(t_\theta^{\ell-1};t_\theta^\ell)$ for

$k \leq \ell \leq n$. Pick $\widetilde{T}$ a solution of

$$\min_{T:\theta(T)=t_\theta^{k-2}} \max_{S:S\cap T=\varnothing;\theta(S)=t_\theta^{k-1}-t_\theta^{k-2}} u(S) = u^\theta(t_\theta^{k-2}; t_\theta^{k-1})$$

As $\theta(\widetilde{T}\cap T_{k-1}) \leq t_\theta^{k-2}$ and $\theta(T_{k-1}) = t_\theta^{k-1}$ we can choose T_{k-2} s.t. $\widetilde{T}\cap T_{k-1} \subseteq T_{k-2} \subseteq T_{k-1}$ and $\theta(T_{k-2}) = t_\theta^{k-2}$. Then we set $R_{k-1} = T_{k-1}\diagdown T_{k-2}$ so that $u(R_{k-1}) \leq u^\theta(t_\theta^{k-2}; t_\theta^{k-1})$ follows from $R_{k-1} \cap \widetilde{T} = \varnothing$ and the definition of $\widetilde{T}$. This completes the induction step. We note finally that each set R^k thus constructed is of θ-size $t_\theta^k - t_\theta^{k-1}$, and that $\max_k u(S_k) = \Gamma^\theta(u;n)$. ■

It is easy to check that no agent can secure more utility than Γ_n^κ in MK_n^κ or Γ_n^θ in $\mathrm{B\&C}_n^\theta$.

Remark 4. The minMax Guarantee and Maxmin upper bound for $u \in \mathcal{M}^\varepsilon(\Omega)$ and $-u \in \mathcal{M}^{-\varepsilon}(\Omega)$, where $\varepsilon = \pm$, are related: $minMax(-u;n) = -Maxmin(u;n)$. With two agents the Guarantees $\Gamma^\kappa(u;2)$ and $\Gamma^\theta(u;2)$ are similarly antisymmetric:

$$\Gamma^\alpha(-u;2) = -\Gamma^\alpha(u;2) \text{ where } \alpha \text{ is } \kappa \text{ or } \theta \tag{8}$$

This is clear for Γ^κ and we check it for Γ^θ by means of the change of variable $S \to S' = \Omega\diagdown S$:

$$\Gamma^\theta(-u;2) = -\min_{0\leq t^1\leq 1} \max\{\min_{\theta(S)=t^1} u(S), \max_{\theta(S)=t^1} u(\Omega\diagdown S)\} =$$

$$-\min_{0\leq t^1\leq 1} \max\{\min_{\theta(S')=1-t^1} u(\Omega\diagdown S'), \max_{\theta(S')=1-t^1} u(S')\}$$

$$= -\min_{0\leq t'\leq 1} \max\{\max_{\theta(S')=t'} u(S'), \min_{\theta(S')=t'} u(\Omega\diagdown S')\}$$

and the claim follows because if two continuous functions $t \to f(t)$ and $t \to g(t)$ intersect in $[0,1]$ and one increases while the other decreases, then $\min_{0\leq t\leq 1} \max\{f(t), g(t)\} = \max_{0\leq t\leq 1} \min\{f(t), g(t\}$.

The identity (8) generalises to $n \geq 3$ for the MK^κ Guarantee, but not for the $B\&C^\theta$ one.

5.3 Microeconomic fair division

We must divide a good manna $\omega \in \mathbb{R}_+^K$ in n shares $z_i \in \mathbb{R}_+^K$. Utilities $u \in \mathcal{M}^+(\omega)$ are continuous and weakly increasing on $[0,\omega]$.

A Moving Knife is a continuous increasing path $t \to K(t)$ from 0 to ω: a natural choice is $K(t) = t\omega, 0 \leq t \leq 1$: the corresponding Guarantee $\Gamma^{\kappa}(u;n) = u(\frac{1}{n}\omega)$ is the *Equal Split* utility $\Gamma^{es}(u;n) = u(\frac{1}{n}\omega)$. A positive, additive measure θ defining B&C$^{\theta}$ is a "price" $\theta(z) = p \cdot z$, $p \in \mathbb{R}^K_+ \diagdown \{0\}$, we write the corresponding Guarantee as Γ^p.

Recall from Section 1 that if an agent's preferences are convex her Equal Split utility equals her *Maxmin* utility, the upper bound on all Fair Guarantees ((1)), in particular weakly larger than the B&C^p guarantee for any p. The converse inequality holds for "concave preferences".

Lemma 5

i) If the upper contours of the utility $u \in \mathcal{M}^+(\omega)$ are convex, then $\Gamma^p(u;n) \leq u(\frac{1}{n}\omega) = Maxmin(u;n)$.

ii) If the lower contours of the utility $u \in \mathcal{M}^+(\omega)$ are convex, then $minMax(u;n) = u(\frac{1}{n}\omega) \leq \Gamma^p(u;n)$.

The equality in statement *i*) was proven in Section 1. A symmetrical argument gives statement *ii*). Pick a hyperplane H supporting the lower contour of u at $\frac{1}{n}\omega$; then the *upper* contour of u at $\frac{1}{n}\omega$ contains one closed half-space cut by H, and every division of the manna includes at least one share in that half-space, implying $u(\frac{1}{n}\omega) \leq minMax(u;n)$, and the reverse inequality is clear.

We turn to a handful of numerical examples where $K = 2$, $\omega = (1,1)$, and $p \cdot z = \frac{1}{2}(x+y)$. Shares are $z = (x,y)$, utilities are 1-homogenous and normalised so that $u(\omega) = 10$. We compute our three Guarantees: Bid and Choose Γ^p, Equal Split, and $minMax$, and compare them to the $Maxmin$ upper bound.

The first three utilities (Leontief, Cobb Douglas and CES) define convex preferences, the last two define "concave preferences" (represented by quadratic and "anti-Leontief" utilities).

Our first table assumes two agents, $n = 2$, and illustrates Lemma 5. An agent with convex (resp. concave) preferences gets a better Guarantee under

Equal Split (resp. Bid and Choose):

$u(x,y)$	$minMax(u;2)$	$\Gamma^p(u;2)$	$u(\frac{1}{2}\omega)$	$Maxmin(u;2)$
$10\min\{x,y\}$	0	3.3	5	5
$10\sqrt{x\cdot y}$	0	4.1	5	5
$\frac{5}{2}(\sqrt{x}+\sqrt{y})^2$	2.5	4.4	5	5
$5(x+y)$	5	5	5	5
$5\sqrt{2(x^2+y^2)}$	5	5.9	5	7.1
$10\max\{x,y\}$	5	6.7	5	10

The equal split partition delivers the $Maxmin$ utility for the first four preferences, and the $minMax$ utilities for the last three. The equi-partition $\Pi = \{(1,0),(0,1)\}$ gives similarly the $minMax$ utilities of the first four, and the $Maxmin$ ones for the last three.

To compute $\Gamma^p(u;2)$ we know from (7) that the optimal bid t^1 (denoted t for simplicity) solves

$$\max_{\frac{1}{2}(x+y)\leq t} u(x,y) = \min_{\frac{1}{2}(x+y)\leq t} u(1-x,1-y) = \min_{\frac{1}{2}(x+y)\geq 1-t} u(x,y)$$

This equality implies $0\leq t\leq \frac{1}{2}$. If u represents convex preferences symmetric in the two goods, $u(x,y)$ is maximal under $\frac{1}{2}(x+y)\leq t$ at $x=y=t$, and minimal under $x+y\geq 2(1-t)$ at $x=1, y=1-2t$. So we must solve $u(t,t)=u(1,1-2t)$: see Figure 2.

If u represents concave symmetric preferences its maximum under $\frac{1}{2}(x+y)\leq t$ is at $x=0, y=2t$, and its minimum under $x+y\geq 2(1-t)$ at $x=y=1-t$, so we solve $u(0,2t)=u(1-t,1-t)$: see Figure 3.

We compute finally the same Guarantees with three agents:

$u(x,y)$	$minMax(u;3)$	$\Gamma^p(u;3)$	$u(\frac{1}{3}\omega)$	$Maxmin(u;3)$
$10\min\{x,y\}$	0	2	3.3	3.3
$10\sqrt{x\cdot y}$	0	2.4	3.3	3.3
$\frac{5}{2}(\sqrt{x}+\sqrt{y})^2$	2	2.5	3.3	3.3
$5(x+y)$	3.3	3.3	3.3	3.3
$5\sqrt{2(x^2+y^2)}$	3.3	4.1	3.3	4.1
$10\max\{x,y\}$	3.3	5	3.3	5

The $minMax$ equi-partition for $u=\frac{5}{2}(\sqrt{x}+\sqrt{y})^2$ and the $Maxmin$ equi-partition for $u'=5\sqrt{2(x^2+y^2)}$ have the same form $\Pi=\{(x,0),(0,x),(1-$

$x, 1-x)\}$: in the former case we find $x = \frac{4}{5}$ and $minMax(u;3) = 2$, in the latter we get $x = 2 - \sqrt{2}$ and $Maxmin(u';3) = 10(\sqrt{2}-1)$. Lemma 5 and the partition $\Pi' = \{(1,0),(0,\frac{1}{2}),(0,\frac{1}{2})\}$ fill the remaining values of $minMax$ and $Maxmin$.

To compute $\Gamma^p(u;3)$ we know by Lemma 4 that the three terms in (*6*) are equal. They are

$u^p(0,t^1) = \max_{\frac{1}{2}(x+y)\leq t^1} u(x,y)$

$u^p(t^1,t^2) = \min_{\frac{1}{2}(x^1+y^1)\leq t^1} \max_{\frac{1}{2}(x+y)\leq t^2-t^1 \text{ and } (x^1+x,y^1+y)\leq(1,1)} u(x,y)$

$u^p(t^2,1) = \min_{\frac{1}{2}(x^2+y^2)\leq t^2} u(1-x^2,1-y^2)$

Clearly $t^1 \leq \frac{1}{3}$ (as $t^2 - t^1 < \frac{1}{3} < t^1$ and $1 - t^2 < \frac{1}{3} < t^1$ are both impossible). Therefore $u^p(0,t^1) = u^p(t^1,t^2)$ is achieved by $t^2 = 2t^1$ (the constraint $(x^1+x, y^1+y) \leq (1,1)$ does not bind). Writing $t = t^1 = t^2 - t^1$ it remains to solve

$$\max_{\frac{1}{2}(x+y)\leq t} u(x,y) = \min_{\frac{1}{2}(x^2+y^2)\leq 2t} u(1-x^2,1-y^2) = \min_{\frac{1}{2}(x+y)\geq 1-2t} u(x,y)$$

When u represents convex preferences symmetric in the two goods, the minimum on the right-hand side is achieved by $(x,y) = (1-4t,1)$ so we solve $u(t,t) = u(1-4t,1)$. See Figure 4.

If u represents concave symmetric preferences, the minimum on the right-hand side is achieved by $(x,y) = (1-2t,1-2t)$ so we solve $u(2t,0) = u(1-2t,1-2t)$. See Figure 5.

6 Concluding comments

Comparing B&C$_n$ versus D&C$_n$ rules The exogenous ordering of the agents greatly affects the outcome of D&C$_n$, whereas B&C$_n$ treats the agents symmetrically. On the other hand the choice of the benchmark measure in B&C$_n$ is exogenous, which allows much, perhaps too much flexibility to the designer.

In D&C$_n$ the dividing agent may have many different strategies guaranteeing her $minMax$ utility. By contrast in B&C$_n$ the solution to programs (7) and (6) is often unique. Multiple choices and the resulting indeterminacy of the outcome may be appealing for the sake of privacy preservation, less so from the implementation viewpoint.

Two challenging open questions 1). Fix the manna $(\Omega, \mathcal{B})$ as in Theorem 1, and the set $[n]$ of agents, each with a utility in $\mathcal{D}(\Omega)$. As explained in Remark 1, Subsection 3.2, Stromquist ([35]) and Su ([36]) showed that an envy-free partition of Ω exists *if all utilities are non negative for all shares.* Is this still true if we remove the sign assumption on utilities?

2) If the utilities vary in a domain $\mathcal{U}(\otimes)$ where the $Maxmin$ utility is not feasible, we would like to discover the family of *undominated* Fair Guarantees $u \to \Gamma(u; n)$. For instance in the microeconomic domain $\mathcal{M}^+(\omega)$ of Subsection 5.3, the Equal Split Guarantee is clearly undominated. We conjecture that in the domains $\mathcal{M}^\pm(\Omega)$ the B&C Guarantees Γ^θ (Subsection 5.2) are undominated as well.

7 Appendices

7.1 Proof of Lemma 1: equi-partition

Recall that we have one fixed agent who wants to split the space into n pieces he regards as all equally valuable. We may assume the value is normalized so that the empty set has value 0 (not that it matters) and by fixing a set of knife cuts, we can think of the goods as being the unit interval $[0, 1)$. Think of the n pieces as n agents who all have the same preference function as the fixed agent.

Take this unit interval and embed it in a curved way in a high dimensional Euclidean space $\mathbb{R}^M$ in such a way that every codimension 1 hyperplane meets the interval in only finitely many points. The standard way to do this is by mapping the point $t \in [0, 1)$ to the point $x(t) = (t, t^2, t^3, \ldots, t^M)$. Any codimension 1 hyperplane is defined by a vector equation $a \cdot x = b$ for some vector $a = (a_1, a_2, \ldots, a_M)$ and some constant b. But then the equation that the point $x(t)$ is on this hyperplane is the polynomial $a_M t^M + \cdots + a_2 t^2 + a_1 t - b = 0$ which has at most M solutions. Thus any codimension 1 hyperplane meets at most M points.

Now we define the configuration space

$$\mathrm{Conf}_n(\mathbb{R}^M) = \left(\prod_{k=1}^{n} \mathbb{R}^M \right) \setminus \{(x_1, x_2, \ldots, x_n) \in (\mathbb{R}^M)^n : x_i = x_j \text{ for some } i \neq j\}.$$

In words the configuration space is just the n-tuples of distinct points chosen from $\mathbb{R}^M$. For any point $(x_1, \ldots, x_n)$ in this configuration space we can assign a division of the interval $[0, 1)$. We simply assign to the k-th agent all points t for which $x(t)$ is closer to his point x_k than to any other agents. There is a slight ambiguity here for points that are equidistant from two agents, but the set of points equidistant from x_j and x_k is the codimension 1 hyperplane that bisects the segment $[x_j, x_k]$ and by our construction these hyperplanes contain only finitely many points. Thus the ambiguity is a finite set (in particular it has measure zero), so we may ignore it. Note that although this description is reasonably pretentious, the actual divisions we get are formed by cutting along a large number of hyperplanes. Thus each agents share will be a disjoint union of finitely many intervals.

For each such division, we can assign a value in $\mathbb{R}^n$ where the k-th coordinate is the utility of the share given to the k-th agent. We want to show that there is some point $(x_1, \ldots, x_n)$ in the configuration space for which this utility lies on the line $(a, a, a, \ldots, a)$, that is, for which every agent's share has the same utility.

Assume this is not the case. Then we can subtract the overall mean from the utilities to get a point in

$$\mathbb{R}^{n-1} = \{(a_1, a_2, \ldots, a_n) : a_1 + a_2 + \cdots + a_n = 0\}$$

and then rescale to get an element of

$$S^{n-2} = \{(a_1, a_2, \ldots, a_n) : a_1 + a_2 + \cdots + a_n = 0 \text{ and } a_1^2 + a_2^2 + \cdots + a_n^2 = 1\}.$$

Thus we have a map

$$f : \mathrm{Conf}_n(\mathbb{R}^M) \to S^{n-2}.$$

Notice that there is a natural action of the symmetric group S_n on the configuration space by permuting the agents and on the sphere by permuting the coordinates and the map f is equivariant under this action. We will show that no such map can exist, in fact we will show that there cannot be such a map which is invariant under the cyclic group $G = \mathbb{Z}/n\mathbb{Z}$ acting by cyclically permuting the agents. From here we use the tools of algebraic topology, as described in ([23]).

If there were such a map, then taking the quotient would give a map

$$g = \overline{f} : \mathrm{Conf}_n(\mathbb{R}^M)/G \to S^{n-2}/G.$$

The action of the cyclic group on S^{n-2} is a free action if n is an odd prime and the quotient is a Lens space, but not in general. In the general the quotient must be interpreted as an orbifold.

Since both quotients have fundamental group G, there are classifying maps from each to the classifying space BG. This classifying space can be thought of as the limit of the quotient $\mathrm{Conf}_n(\mathbb{R}^M)/G$ as M goes the infinity, and hence we may choose M large enough that the homologies agree as far as we desire. Since the universal cover of S^{n-2}/G is the $(n-2)$-sphere, we can think of the classifying space BG as being built from this $(n-2)$-sphere by adding cells of dimension $(n-1)$ and higher and the classifying map $h : S^{n-2}/G \to BG$ as being an inclusion. In particular, this means that

$$h_* : H_k(S^{n-2}/G; \mathbb{Z}) \to H_k(BG; Z)$$

is an isomorphism for $k < n-2$ and a surjection for $k = n-2$. For n odd, the action of G is by orientation-preserving maps and therefore S^{n-2}/G has a fundamental class and

$$h_* : H_{n-2}(S^{n-2}/G; \mathbb{Z}) \cong \mathbb{Z} \to H_{n-2}(BG; Z) = \mathbb{Z}/n\mathbb{Z}$$

is the usual projection. If n is even, then the action is by orientation-reversing maps and both $H_{n-2}(S^{n-2}/G; \mathbb{Z})$ and $H_{n-2}(BG; Z)$ vanish. In either case by the universal coefficent theorem we find that

$$h^* : H^k(BG, \mathbb{Z}/n\mathbb{Z}) \to H^k(S^{n-2}/G; \mathbb{Z}/n\mathbb{Z})$$

is an isomorphism for $0 \leq k \leq n-2$.

Recall that the cohomology ring

$$H^*(B\mathbb{Z}/n\mathbb{Z}; \mathbb{Z}/n\mathbb{Z}) = \mathbb{Z}/n\mathbb{Z}[\alpha, \beta]/(\beta^2 = k\alpha)$$

is generated by two elements β of dimension 1 and α of dimension 2 with the relation that if n is odd then $\beta^2 = 0$ (so $k = 0$ in the formula above) and if $n = 2k$ is even, then $\beta^2 = k\alpha$. Since the map $S^{n-2}/G \to B\mathbb{Z}/n\mathbb{Z}$ is an isomorphism on the fundamental group and hence on first homology, it follows from the universal coefficients theorem that h^* is an isomorphism on both H^1 (which comes from $Hom(H_1)$) and on H^2 (which comes from $Ext(H_1)$). Hence $H^*(S^{n-2}/G; \mathbb{Z}/n\mathbb{Z})$ is generated by the images of α and

β (which again we will denote by the same symbols), with just the added restriction that all product of dimension over $n-2$ vanish.

Now we have a map on cohomology rings

$$g^* : H^*(S^{n-2}/G, \mathbb{Z}/n\mathbb{Z}) \to H^*(\mathrm{Conf}_n(\mathbb{R}^M)/G, \mathbb{Z}/n\mathbb{Z}).$$

This map is the identity map on the fundamental group, hence on H_1, and hence g^* is the identity on $H^1$and H^2. Thus the elements α and β must map to the analogous elements for the configuration space. But this cannot happen since the products of α and β of dimension more than $n-2$ vanish on the left hand side, but (for M large enough) not on the right hand side. Thus we have a contradiction.

7.2 Proof of Lemma 4

1). *First statement.* Recall that we can replace in definition (*5*) the equalities like $\theta(T) = t^1$ with inequalities $\theta(T) \leq t^1$. We check first that the correspondence $t \to \{S \in \mathcal{B} | \theta(S) \leq t\}$ is continuous. Upper hemi continuity follows by the continuity of θ. For lower hemi continuity pick a sequence t_n converging to t and $S \in \mathcal{B}$ s.t. $\theta(S) \leq t$. If t_n has a decreasing subsequence, we set $S_n = S$ so that $\theta(S_n) \leq t_n$ and S_n converges to S. If t_n has an increasing subsequence we construct an inclusion increasing sequence S_m converging to S and s.t. $|S_m| < |S|$ for all m: because θ increases strictly, so does the sequence $\theta(S_m)$ converging to $\theta(S)$, therefore we can pick subsequences S_p of S_m and t_p of t_n s.t. $\theta(S_p) \leq t_p$, as desired.

Next we apply the Maximum Theorem twice. The first one to show that the function $(T, t^1, t^2) \to C(T, t^1, t^2) = \max\{u(S) | S \subset \Omega \diagdown T; \theta(T \cup S) \leq t^1 + t^2\}$ is continuous because the correspondence $(T, t^1, t^2) \to \{S | S \subset \Omega \diagdown T; \theta(T \cup S) \leq t^1 + t^2\}$ is continuous. The second one to deduce that the function $\min_{T:\theta(T) \leq t^1} C(T, t^1, t^2)$ is continuous.

2). *Second statement.* For simplicity we assume $n = 3$, the general proof is entirely similar. Fixing u and t^1 there is some t^2 such that $u^\theta(t^1; t^2) = u^\theta(t^2; 1)$. This is because of the monotonicity properties of u^θ and of the inequalities $u^\theta(t^1; t^1) = 0 \leq u^\theta(t^1; 1)$ and $u^\theta(t^1; 1) \geq 0 = u^\theta(1; 1)$. This common value is unique (though t^2 may not be) and defines a function $g(t^1) = u^\theta(t^1; t^2) = u^\theta(t^2; 1)$. It is easy to check from the continuity and monotonicity properties of u^θ that g is weakly decreasing and continuous. Then we find in the same way t^1 s.t. $g(t^1) = u^\theta(0; t^1)$.

Check finally that if $\tau_* \in \Upsilon(n)$ is such that all terms $u^\theta(t_*^k; t_*^{k+1})$, $0 \leq k \leq n-1$, equal a common value λ, then τ_* solves program (6). If it does not there is a τ such that $u^\theta(t^k; t^{l+1}) > \lambda$ for $0 \leq k \leq n-1$. Applying this inequality at $k=0$ gives $t^1 > t_*^1$; next at $k=1$ we get $u^\theta(t^1, t^2) > u^\theta(t_*^1, t_*^2)$ implying $t^2 > t_*^2$; and so on until we reach a contradiction with the fact that both τ and τ_* are in $\Upsilon(n)$.

Finally, the optimal sequence t^k increases in k, strictly if u is not everywhere zero because $u(t,t) = 0$ for all t.

References

[1] Aigner-Horev E, Segal-Halevi E. 2019. Envy-free Matchings in Bipartite Graphs and their Applications to Fair Division, arXiv.org/cs/arXiv:1901.09527

[2] Aumann Y, Dombb Y. 2010. The Efficiency of Fair Division with Connected Pieces, *International Workshop on Internet and Network Economics WINE 2010*: Internet and Network Economics, 26-37

[3] Avvakumov s, Karasev R. 2019. Envy-free Division under Mapping Degree, arxiv.org/pdf/1907.11183.pdf

[4] Aziz H, Caragianis I, Igarashi A, Walsh T. 2018. Fair allocation of combinations of indivisible goods and chores, arXiv.org > cs > arXiv:1807.10684

[5] Aziz A, McKenzie S. 2016. A Discrete and Bounded Envy-free Cake Cutting Protocol for Any Number of Agents, *IEEE 57th Annual Symposium on Foundations of Computer Science (FOCS)*, New Brunswick, NJ, USA, 9-11 Oct. 2016

[6] Barbera S, Dutta B. 1982. Implementability via protective equilibria. *Journal of Mathematical Economics*, 10 (1), p. 49-65

[7] Berliant M, Dunz K, Thomson W. 1992. On the fair division of a heterogeneous commodity, *J. Math. Econ.*, 21, 201–16

[8] Bogomolnaia A, Moulin H. 2004. Random matching under divhotomous preferences, Econometrica, 72, 1, 257-279

[9] Bogomolnaia A, Moulin H, Sandomirskiy F, Yanovskaia E. 2019. Dividing Bads under Additive Utilities, *Soc.Choice Welfare*, 52 (3), 395-417

[10] Bogomolnaia A, Moulin H, Sandomirskiy F, Yanovskaia E. 2017. Competitive division of a mixed manna, *Econometrica*, 85(6), 1847-71

[11] Bouveret S, Lemaitre M. 2016. Characterizing conflicts in fair division of indivisiblegoods using a scale of criteria. Auton Agent Multi-Agent Syst, 30:259–290

[12] Brams SJ, Taylor AD. 1995. An envy-free cake division protocol, *American Math. Monthly*, 102(1), 9-18

[13] Brams SJ, Taylor AD. 1996. *Fair Division: From Cake-Cutting to Dispute Resolution*, Cambridge University Press, Cambridge, Mass, USA

[14] Branzei S. 2015. A note on envy-free cake cutting with polynomial valuations, *Information Processing Letters* 115, 2, 93–95

[15] Budish E. 2011. The combinatorial assignment problem: Approximate competitive equilibrium from equal incomes. *J. Polit. Econ.* 119, 6, 1061–103

[16] Cseh A, Fleiner T. 2018. The complexity of cake cutting with unequal shares, arxiv.org/cs/1709.03152

[17] Crew L, Narayanan B, Spirkl S. 2019. Disproportionate division, arXiv.org > math > arXiv:1909.07141

[18] Dubins LE, Spanier EH. 1961. How to cut a cake fairly. *American Math. Monthly*, 68, 1–17

[19] Kuhn, H.1967. On games of fair division, *Essays in Mathematical Economics in Honour of Oskar Morgenstern*, Princeton University Press, pp. 29–37

[20] Kurokawa D, Lai JK., Procaccia AD. 2013. How to Cut a Cake Before the Party Ends, Proceedings of the 27th AAAI Conference on Artificial Intelligence

[21] Kurokawa D, Procaccia AD, Wang J. 2016. When can the maximin share guarantee be guaranteed?, *Proceedings of the 30th AAAI Conference on Artificial Intelligence (AAAI),*523-9

[22] Lovasz L, Plummer MD. 1986. *Matching Theory*, Annals of Discrete Mathematics, Vol. 29. Amsterdam: North-Holland.

[23] Massey WS. 1977. *Algebraic Topology: An Introduction*, Springer.

[24] Meunier F, Zerbib S. 2018. Envy-free cake division without assuming the players prefer nonempty pieces, forthcoming, *Israel Journal of Mathematics*, arxiv.org/pdf/1804.00449.pdf

[25] Moulin H. 1992. All Sorry to Disagree: a General Principle for the Provision of Non-rival Goods, *Scandinavian Journal of Economics*, 94, 37–51

[26] Moulin H. 1992. Welfare Bounds in the Cooperative Production Problem, Games and Economic Behavior, 4, 373–401

[27] Moulin H. 2019. Fair Division in the Internet Age, Annual Review of Economics,11:407-41.

[28] Munkres JR. 1996. *Elements of Algebraic Topology*, Avalon.

[29] Procaccia AD. 2013. Cake cutting: not just child's play, *Communications of the ACM (CACM Homepage archive)*, 56(7), 78-87

[30] Procaccia AD, Wang J. 2014. Fair Enough: Guaranteeing Approximate Maximin Shares, *Proceedings of the 14th ACM Conference on Economics and Computation EC'14*, 675–692

[31] Robertson JM, Webb WA.1998. *Cake Cutting Algorithms: Be Fair If You Can*, A. K. Peters

[32] Segal-Halevi E. 2018. Fairly Dividing a Cake after Some Parts Were Burnt in the Oven, *Proceeding of AAMAS '18,* 1276-1284

[33] Segal-Halevi E, Nitzan S, Hassidim A, Aumann Y. 2017. Fair and square: Cake-cutting in two dimensions, *J. Math. Econ.*, 70, 1-289

[34] Steinhaus H. 1949. Sur la division pragmatique Econometrica (supplement), 17, 315-319

[35] Stromquist W. 1980. How to cut a cake fairly, *Amer. Math. Monthly* 87, 640-644

[36] Su F. E. 1999. Rental harmony: Sperner's lemma in fair division. *American Math. Monthly,* 10, 930–42

[37] Thomson W. 2005. *The theory of fair allocation.* Princeton University Press, Princeton NJ, USA

[38] Woodall DR. 1980. Dividing a cake fairly, *Journal of Mathematical Analysis and Applications*, 78(1):233-247