A lower bound on seller revenue in single buyer monopoly auctions

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Abstract

We consider a monopoly seller who optimally auctions a single object to a single potential buyer, with a known distribution of valuations. We show that a tight lower bound on the seller's expected revenue is $1/e$ times the geometric expectation of the buyer's valuation, and that this bound is uniquely achieved for the equal revenue distribution. We show also that when the valuation's expectation and geometric expectation are close, then the seller's expected revenue is close to the expected valuation.

1 Introduction

Consider a monopoly seller, selling a single object to a single potential buyer. We assume that the buyer has a valuation for the object which is unknown to the seller, and that the seller's uncertainty is quantified by a probability distribution, from which it believes the buyer picks its valuation.

Assuming that the seller wishes to maximize its expected revenue, Myerson [1] shows that the optimal incentive compatible mechanism involves a simple one-time offer: the seller (optimally) chooses a price and offers the buyer to buy the object for this price; the assumption is that the buyer accepts the offer if its valuation exceeds this price. Myerson's seminal paper has become a classical result in auction theory, with numerous follow-up studies. A survey of this literature is beyond the scope of this paper (see, e.g., [2, 3]).

The expected seller revenue is an important, simple intrinsic characteristic of the valuation distribution. A natural question is its relation with various other properties of the distribution. For example, can seller revenue be bounded given such characterizations of the valuation as its expectation, entropy, etc.? An immediate upper bound on seller revenue is the buyer's expected valuation. In fact, the seller can extract the buyer's expected valuation only if the seller knows the buyer's valuation exactly - i.e., the distribution over valuations is a point mass.

Lower bounds on seller revenue are important in the study of approximations to Myerson auctions (see., e.g., Hartline and Roughgarden [4], Daskalakis and Pierrakos [5]). A general lower bound on the seller's revenue is known when the distribution of the buyer's valuation has a monotone hazard rate; in this case, the seller's expected revenue is at least $1/e$ times the expected valuation (see Hartline, Mirrokni and Sundararajan [6], as well as Dhangwatnotai, Roughgarden and Yan [7]).

This bound does not hold in general: as an extreme example, the equal revenue distribution discussed below has infinite expectation but finite seller revenue. The family of monotone hazard rate distributions does not include many important distributions such the Pareto distribution or other power law distributions, or in fact any distribution that doesn't have a very thin tail, vanishing at least exponentially. The above mentioned lower bound for monotone hazard rate distributions

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does not apply to these distributions, and indeed it seems that the literature lacks any similar, general lower bounds on seller revenue.

The *geometric expectation* of a positive random variable X is $\mathbb{G}[X] = \exp(\mathbb{E}[\log X])$ (see, e.g., [8]). We show that a general lower bound on the seller's expected revenue is $1/e$ times the geometric expectation of the valuation. Equivalently, the (natural) logarithm of the expected seller revenue is greater than or equal to the expectation of the logarithm of the valuation, minus one. This bound holds for any distribution of positive valuations. Notably, the *regularity* condition, which often appears in the context of Myerson auctions, is not required here. This result is a new and perhaps unexpected connection between two natural properties of distributions: the geometric expectation and expected seller revenue.

We show that this bound is tight in the following sense: for a fixed value of the geometric mean, there is a unique cumulative distribution function (CDF) of the buyer's valuation for which the bound is achieved; this distribution is the equal revenue distribution, with CDF of the form $F(v) = 1 - c/v$ for $v \geq c$. This distribution is "special" in the context of single buyer Myerson auctions, as it is the only one where seller revenue is identical for all prices.

The ratio between expected valuation and expected seller revenue is a natural measure of the uncertainty of the valuation distribution. Also, the discrepancy between the geometric expectation and the (arithmetic) expectation of a positive random variable is a well known measure of its dispersion. Hence, when the ratio between the expectations is close to one, one would expect the amount of uncertainty to be low and therefore seller revenue to be close to the expected valuation. We show that this is indeed the case: when the buyer's valuation has finite expectation, and the geometric expectation is within a factor of $1-\delta$ of the expectation, then seller revenue is within a factor of $1 - 2^{4/3}\delta^{1/3}$ of the expected valuation. Similarly, it is easy to show that when the variance of the valuation approaches zero then seller revenue also approaches the expected valuation.

2 Definitions and results

We consider a seller who wishes to sell a single object to a single potential buyer. The buyer has a valuation V for the object which is picked from a distribution with CDF F, i.e. $F(v) = \mathbb{P}[V \leq v]$.

We assume that V is positive, so that $\mathbb{P}[V \leq 0] = 0$ or $F(0) = 0$. We otherwise make no assumptions on the distribution of V; it may be atomic or non-atomic, have or not have an expectation, etc.

The seller offers the object to the buyer for a fixed price p. The buyer accepts the offer if $p < V$, in which case the seller's revenue is p. Otherwise, i.e., if $p \geq V$, then the seller's revenue is 0. Thus, the seller's expected revenue for price p, which we denote by $\mathbb{U}_p[V]$, is given by

$$\mathbb{U}_p[V] = p\mathbb{P}[p < V] = p(1 - F(p)). \tag{1}$$

We define

$$\mathbb{U}[V] = \sup_p \mathbb{U}_p[V] = \sup_p p(1 - F(p)). \tag{2}$$

When this supremum is achieved for some price p then $\mathbb{U}[V]$ is the seller's maximal expected revenue, achieved in the optimal Myerson auction with price p.

We define the *geometric expectation* (see, e.g., [8]) of a positive real random variable X by $\mathbb{G}[X] = \exp(\mathbb{E}[\log X])$. Note that $\mathbb{G}[X] \leq \mathbb{E}[X]$ by Jensen's inequality, and that equality is achieved only for point mass distributions, i.e., when the buyer's valuation is some fixed number. Note that likewise $\mathbb{U}[V] \leq \mathbb{E}[V]$, again with equality only for point mass distributions.

The equal revenue distribution with parameter c has the following CDF:

$$\Phi_c(p) = \begin{cases} 0 & p \leq c \\ 1 - \frac{c}{p} & p > c \end{cases}. \tag{3}$$

It is called "equal revenue" because if V_c has CDF Φ_c then $\mathbb{U}_p[V_c] = \mathbb{U}[V_c]$ for all $p \geq c$.

Our main result is the following theorem.

Theorem 2.1. *Let V be a positive random variable. Then $\mathbb{U}[V] \geq \frac{1}{e}\mathbb{G}[V]$, with equality if and only if V has the equal revenue CDF Φ_c with $c = \mathbb{U}[V]$.*

Proof. Let V be a positive random variable with CDF F. By Eq. 2 we have that

$$\log \mathbb{U}[V] \geq \log p + \log(1 - F(p)) \tag{4}$$

for all p. We now take the expectation of both sides with respect to $p \sim F$:

$$\int_0^\infty \log \mathbb{U}[V]\, dF(p) \geq \int_0^\infty \log p\, dF(p) + \int_0^\infty \log(1 - F(p)) dF(p). \tag{5}$$

Since $\mathbb{U}[V]$ is a constant then the l.h.s. equals $\log \mathbb{U}[V]$. The first addend on the r.h.s. is simply $\mathbb{E}[\log V]$. The second is $\mathbb{E}[\log(1 - F(V))]$; note that $F(V)$ is distributed uniformly on $[0, 1]$, and that therefore

$$\mathbb{E}[\log(1 - F(V))] = \int_0^1 \log(1 - x) dx = -1.$$

Hence Eq. 5 becomes:

$$\log \mathbb{U}[V] \geq \mathbb{E}[\log V] - 1,$$

and

$$\mathbb{U}[V] \geq \frac{1}{e} \exp(\mathbb{E}[\log V]) = \frac{1}{e}\mathbb{G}[V].$$

To see that $\mathbb{U}[V] = \frac{1}{e}\mathbb{G}[V]$ only for the equal revenue distribution with parameter $\mathbb{U}[V]$, note that we have equality in Eq. 4 for all p in the support of F if and only if $F = \Phi_c$ for some c, and that therefore we have equality in Eq. 5 if and only if $F = \Phi_c$ for some c. Finally, a simple calculation yields that $c = \mathbb{U}[V]$. □

Note that this proof in fact demonstrates a stronger statement, namely that the expected revenue is at least $\frac{1}{e}\mathbb{G}[V]$ for a seller picking a random price from the distribution of V. Dhangwatnotai, Roughgarden and Yan [7] use similar ideas to show lower bounds on revenue, for valuation distributions with monotone hazard rates.

We next show that when the geometric expectation approaches the (arithmetic) expectation then the seller revenue also approaches the expectation.

Theorem 2.2. *Let V be a positive random variable with finite expectation, and let $\mathbb{G}[V] = (1 - \delta)\mathbb{E}[V]$. Then $\mathbb{U}[V] \geq \left(1 - 2^{4/3}\delta^{1/3}\right)\mathbb{E}[V]$.*

Proof. Let V be a positive random variable with finite expectation, and denote $1-\delta = \frac{\mathbb{G}[V]}{\mathbb{E}[V]}$. We normalize V so that $\mathbb{E}[V]=1$, and prove the claim by showing that $\mathbb{U}[V] \geq 1 - 2^{4/3}\delta^{1/3}$.

Consider the random variable $V-1-\log V$. Since $\mathbb{E}[V]=1$, we have that $\mathbb{E}[V-1-\log V] = -\log \mathbb{G}[V] = -\log(1-\delta)$. Since $x-1 \geq \log x$ for all $x>0$, then $V-1-\log V$ is non-negative. Hence by Markov's inequality

$$\mathbb{P}\left[V-1-\log V \geq -k\log(1-\delta)\right] \leq \frac{1}{k},$$

or

$$\mathbb{P}\left[Ve^{1-V} \leq (1-\delta)^k\right] \leq \frac{1}{k}. \tag{6}$$

This inequality is a concentration result, showing that when δ is small then Ve^{1-V} is unlikely to be much less than one. However, for our end we require a concentration result on V rather than on Ve^{1-V}; that will enable us to show that the seller can sell with high probability for a price close to the arithmetic expectation. To this end, we will use the *Lambert W function*, which is defined at x as the solution of the equation $W(x)e^{W(x)} = x$. We use it to solve the inequality of Eq. 6 and arrive at

$$\mathbb{P}\left[V \leq -W\left(-(1-\delta)^k/e\right)\right] \leq \frac{1}{k},$$

which is the concentration result we needed: V is unlikely to be small when δ is small. It follows that by setting the price at $-W\left(-(1-\delta)^k/e\right)$, the seller sells with probability at least $1-1/k$, and so

$$\mathbb{U}[V] \geq -W\left(-(1-\delta)^k/e\right) \cdot \left(1-1/k\right).$$

Now, an upper bound on W is the following [9]:

$$W(x) \leq -1 + \sqrt{2(ex+1)},$$

and so

$$\mathbb{U}[V] \geq \left(1-\sqrt{2(1-(1-\delta)^k}\right) \cdot \left(1-1/k\right) \geq \left(1-\sqrt{2\delta k}\right) \cdot \left(1-1/k\right).$$

Setting $k=(2\delta)^{-1/3}$ we get

$$\mathbb{U}[V] \geq \left(1-(2\delta)^{1/2}(2\delta)^{-1/6}\right) \cdot \left(1-(2\delta)^{1/3}\right) \geq 1-2(2\delta)^{1/3}.$$

□

3 Open questions

It may very well be possible to show tighter *upper* bounds for $\mathbb{U}[V]$, using continuous entropy. For example, let V have expectation 1 and entropy at least 1. Then $\mathbb{U}[V]$ is at most $1/e$: in fact, it is equal to $1/e$ since, by maximum entropy arguments, there is only one distribution on $\mathbb{R}^+$ (the exponential with expectation 1) that satisfies both conditions, and for this distribution $\mathbb{U}[V]=1/e$.

One could hope that it is likewise possible to prove upper bounds on $\mathbb{U}[V]$, given that V has expectation 1 and entropy at least $h<1$; intuitively, the entropy constraint should force V to spread rather than concentrate around its expectation, decreasing the seller's expected revenue.

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