CHOQUET-DENY GROUPS AND THE INFINITE CONJUGACY CLASS PROPERTY

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Abstract. A countable discrete group \( G \) is Choquet-Deny if it has a trivial Poisson boundary for every non-degenerate probability measure. We show that a finitely generated group \( G \) is Choquet-Deny if and only if it is virtually nilpotent. For general countable discrete groups, we show that \( G \) is Choquet-Deny if and only if none of its quotients have the infinite conjugacy class property. Moreover, when \( G \) is not Choquet-Deny, then there exists a symmetric, finite entropy non-degenerate measure that has a non-trivial Poisson boundary.

1. Introduction

Let \( G \) be a countable discrete group. A probability measure \( \mu \) on \( G \) is non-degenerate if its support generates \( G \) as a semigroup.\(^1\) A function \( f: G \to \mathbb{R} \) is \( \mu \)-harmonic if \( f(k) = \sum_{g \in G} \mu(g)f(kg) \) for all \( k \in G \). We say that the measured group \((G, \mu)\) is Liouville if all the bounded \( \mu \)-harmonic functions are constant; this is equivalent to the triviality of the Poisson boundary \( \Pi(G, \mu) \) \(^{13–15}\) (also called the Furstenberg-Poisson boundary; for formal definitions see also, e.g., Furstenberg and Glasner \(^{12}\), Bader and Shalom \(^1\), or a survey by Furman \(^{11}\)).

When \( G \) is non-amenable, \((G, \mu)\) is not Liouville for every non-degenerate \( \mu \) \(^{15}\). Conversely, when \( G \) is amenable, then there exists some non-degenerate \( \mu \) such that \((G, \mu)\) is Liouville, as has been shown by Kaimanovich and Vershik \(^{23}\) and Rosenblatt \(^{25}\). It is natural to ask for which groups \( G \) it holds that \((G, \mu)\) is Liouville for every non-degenerate \( \mu \). We call such groups Choquet-Deny groups; as we discuss in §1.1, there are a few variants of this definition (see, e.g., \(^{16–18}\), or \(^{20}\)), which, however, we show to be equivalent.\(^2\)

\(^{1}\)In the context of Markov chains such measures are called irreducible.

\(^{2}\)A related concept, which we will not study, is that of a Liouville group (see, e.g., \(^{4, 22}\)), which usually is taken to mean that \((G, \mu)\) is Liouville for every symmetric, finitely supported \( \mu \).
The classical Choquet-Deny Theorem (which was first proved for $\mathbb{Z}^d$ by Blackwell [3]) states that abelian groups are Choquet-Deny [5]; the same holds for virtually nilpotent groups [7]. There are many examples of amenable groups that are not Choquet-Deny: first examples of countable discrete groups\(^3\) are due to Kaimanovich [21] and Kaimanovich and Vershik [23], including locally finite groups; Erschler shows that finitely generated solvable groups that are not virtually nilpotent are not Choquet-Deny [9], and that even some groups of intermediate growth are not Choquet-Deny [8]. Kaimanovich and Vershik [23, p. 466] conjecture that “Given an exponential group $G$, there exists a symmetric (nonfinitary, in general) measure with non-trivial boundary.” See Bartholdi and Erschler [2] for additional related results and further references and discussion.

Our main result is a characterization of Choquet-Deny groups. We say that $G$ has the infinite conjugacy class property (ICC) if it is non-trivial, and if each of its non-trivial elements has an infinite conjugacy class. We say that $\mu$ is fully supported if $\text{supp} \, \mu = G$; obviously this implies that $\mu$ is non-degenerate.

**Theorem 1.** A countable discrete group $G$ is Choquet-Deny if and only if it has no ICC quotients. Moreover, when $G$ does have an ICC quotient, then there exists a fully supported, symmetric, finite entropy probability measure $\mu$ on $G$ such that $(G, \mu)$ is not Liouville. In particular, if $G$ is finitely generated, then it is Choquet-Deny if and only if it is virtually nilpotent.

The case of a group with no ICC quotients was shown by Jaworski [19, Theorem 4.8].\(^4\) Our contribution is therefore in the proof of the other direction, which appears in §2.

The implication for finitely generated groups is a consequence of the fact that in this class, virtually nilpotent groups are precisely those with no ICC quotients (see [24, Theorem 2] and [6, Theorem 2]). Since exponential groups are not virtually nilpotent, Theorem 1 implies that the above mentioned conjecture of Kaimanovich and Vershik [23] is correct.

A very recent result by three of the authors of this paper shows that a countable discrete group is strongly amenable if and only if it has no ICC quotients [10]. This implies that $G$ is strongly amenable if and only if $(G, \mu)$ is Liouville for every non-degenerate $\mu$, paralleling the above mentioned characterization of amenability as equivalent to the

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\(^3\)In the Lie group setting, an example of an amenable group that is not Choquet-Deny was already known to Furstenberg [13].

\(^4\)In fact, Jaworski proves there a stronger statement; see the discussion in §1.1.
existence of such a $\mu$. While the proofs of these two similar results are different, it is natural to ask whether there is some deeper connection between strong amenability and the Choquet-Deny property.

1.1. Different possible definitions of Choquet-Deny groups.
Our definition of Choquet-Deny groups is not the usual one, which states that a group is Choquet-Deny if $(G, \mu)$ is Liouville for every adapted measure $\mu$, where $\mu$ is adapted if its support generates $G$ as a group (rather than as a semigroup, as in the non-degenerate case) [16–18]. Yet another definition used in the literature requires that for every $\mu$, every bounded $\mu$-harmonic function is constant on the left cosets of $G_\mu$, where $G_\mu$ is the group generated by the support of $\mu$ [20].

While a priori these are different definitions, they are equivalent, as demonstrated by our result and by Jaworski’s Theorem 4.8 in [19]. Jaworski shows that groups with no ICC quotients are Choquet-Deny according to all of these notions. Since our construction of $\mu$ with a non-trivial boundary yields measures that are supported on all of $G$ (hence non-degenerate, hence adapted), it shows that groups with ICC quotients are not Choquet-Deny according to these definitions. Indeed, our result shows that the class of Choquet-Deny groups (whether defined with adapted or with non-degenerate measures) is closed to taking subgroups, which, to the best of our knowledge, was also not previously known.

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2. Proofs

In this section we prove the main result of our paper, Theorem 1. Unless otherwise states, we will assume that all groups are countable and discrete.

Recall that a probability measure $\mu$ on $G$ is symmetric if $\mu(g) = \mu(g^{-1})$ for all $g \in G$. Its Shannon entropy (or just entropy) is $H(\mu) = -\sum_{g \in G} \mu(g) \log \mu(g)$.

Our Theorem 1 is a direct consequence of [19, Theorem 4.8], which proves it for the case of groups with no ICC quotients, and of the following proposition, which handles the case of groups with ICC quotients.

Proposition 2.1. Let $G$ be a group with an ICC quotient. Then there exists a fully-supported, symmetric, finite entropy probability measure $\mu$ on $G$ such that $\Pi(G, \mu)$ is non-trivial.
The main technical effort in the proof of Proposition 2.1 is in the proof of the following proposition.

**Proposition 2.2.** Let \( G \) be an amenable ICC group. There exists a fully supported, symmetric, finite entropy probability measure \( \mu \) on \( G \) and an element of the group \( h \in G \), such that

\[
\lim_{m \to \infty} \|h\mu^m - \mu^m\| > 0.
\]

Here \( \mu^m \) is the \( m \)-fold convolution \( \mu * \cdots * \mu \). We will prove this Proposition later, and now turn to the proof of Proposition 2.1.

**Proof of Proposition 2.1.** The case of non-amenable \( G \) is known, so assume that \( G \) is amenable and has an ICC quotient \( Q \). Applying Proposition 2.2 to \( Q \) yields a finite entropy, symmetric measure \( \tilde{\mu} \) on \( Q \) that is fully supported, and satisfies (2.1) for some \( h \in Q \).

Since \( \tilde{\mu} \) has full support and satisfies (2.1), it follows from [16, Theorem 2] that \( (Q, \tilde{\mu}) \) has a non-trivial Poisson boundary. Let \( \mu \) be any symmetric, finite entropy non-degenerate probability measure on \( G \) that is projected to \( \tilde{\mu} \); the existence of such a \( \mu \) is straightforward. Then \( (G, \mu) \) has a non-trivial Poisson boundary. \( \square \)

### 2.1. Switching Elements

Here we introduce two notions: switching elements and super-switching elements. We will use these notions in the proof of Proposition 2.2.

**Definition 2.3.** Let \( X \) be a finite symmetric subset of a group \( G \). We denote \( \tilde{X} = X \setminus \{e\} \).

- We call \( g \in G \) a **switching element for** \( X \) if
  \[
  X \cap g\tilde{X}g^{-1} = \emptyset.
  \]

- We call \( g \in G \) a **super-switching element for** \( X \) if
  \[
  X \cap (g\tilde{X}g \cup g\tilde{X}g^{-1} \cup \tilde{X}g \cup g^{-1}\tilde{X}g^{-1}) = \emptyset.
  \]

Note that since \( X \) is symmetric, for \( g \in G \) to be super-switching for \( X \), we just need to have that \( g \) is switching for \( X \), and that \( X \cap g\tilde{X}g = \emptyset \).

**Proposition 2.4.** Let \( G \) be a discrete (not necessarily countable) amenable ICC group, and let \( X \) be a finite symmetric subset of \( G \). The set of super-switching elements for \( X \) is infinite.

**Proof of Proposition 2.4.** Fix an invariant finitely additive probability measure \( d \) on \( G \). For \( A \subseteq G \), we call \( d(A) \) the density of \( A \). We will need the fact that infinite index subgroups have zero density.
Let $C_G(x)$ be the centralizer of a non-identity $x \in X$. Then, since $X$ is finite, there is a finite set of cosets of $C_G(x)$ that includes all $g \in G$ such that $g^{-1}xg \in X$. So, non-switching elements for $X$ are in the union of finitely many cosets of subgroups with infinite index, since $G$ is ICC. This means that the set of non-switching elements for $X$ has zero density, and so the set of switching elements for $X$ has density one, as does the set $S$ of elements $g \in G$ such that both $g$ and $g^{-1}$ are switching for $X$.

Let $T$ be the set of all super-switching elements for $X$. Let $A \subseteq G$ be the set of involutions $\{g \in G \mid g^2 = e\}$. If $d(A) > 0$, then $d(A \cap S) > 0$. On the other hand, for any $g \in A \cap S$, since $g$ is switching for $X$ and $g^{-1} = g$, $g$ is super-switching for $X$. Hence $A \cap S \subseteq T$. This shows that if $d(A) > 0$, then $d(T) \geq d(A \cap S) > 0$, and so we are done.

So, we can assume that $d(A) = 0$. For any $x \in X \setminus \{e\}$ and $y \in X$, let $S_{x,y} = \{g \in S \mid gxg = y\}$. Note that

$$T = S \setminus \bigcup_{x \in X \setminus \{e\}} S_{x,y}.$$ 

It is thus enough to be shown that each $S_{x,y}$ has zero density. So assume for the sake of contradiction that $d(S_{x,y}) > 0$. Fix $g \in S_{x,y}$. We have the following for all $h \in g^{-1}S_{x,y}$.

$$gxg = y = ghxgh \implies (xg) = h(xg)h$$
$$\implies (xg)^{-1}h^{-1}(xg) = h$$
$$\implies h = (xg)^{-1}h^{-1}(xg)$$
$$= (xg)^{-1}[h(xg)^{-1}h^{-1}(xg)]^{-1}(xg)$$
$$= (xg)^{-2}h(xg)^2$$
$$\implies h \text{ is in the centralizer of } (xg)^2.$$ 

So, the centralizer of $(xg)^2$ includes $g^{-1}S_{x,y}$, which has a positive density. So, the centralizer of $(xg)^2$ has finite index. This implies that $(xg)^2 = e$, since $G$ is ICC, and so only the identity can have a finite index centralizer. Hence $xg \in A$ for all $g \in S_{x,y}$. So $xS_{x,y} \subseteq A$. Hence $S_{x,y}$ also has zero density, which is a contradiction. 

2.2. A Heavy-Tailed Probability Distribution on $\mathbb{N}$. Here we state and prove a lemma about the existence of a probability distribution on $\mathbb{N}$ such that infinite i.i.d. samples from this measure have certain properties. We will use this distribution in the proof of Proposition 2.2.

**Lemma 2.5.** Let $p$ be the following probability measure on $\mathbb{N}$: $p(n) = cn^{-5/4}$, where $1/c = \sum_{n=1}^{\infty} n^{-5/4}$. Then $p$ has finite entropy and the
following property: for any \( \varepsilon > 0 \) there exist constants \( K_\varepsilon, N_\varepsilon \in \mathbb{N} \) such that for any natural number \( m \geq K_\varepsilon \) there exists an \( E_{\varepsilon, m} \subseteq \mathbb{N}^m \) such that:

1. \( p^{\times m}(E_{\varepsilon, m}) \geq 1 - \varepsilon \), where \( p^{\times m} \) is the \( m \)-fold product measure \( p \times \cdots \times p \).
2. For any \( s = (s_1, \ldots, s_m) \in E_{\varepsilon, m} \), the maximum of \( \{s_1, \ldots, s_{K_\varepsilon}\} \) is at most \( N_\varepsilon \).
3. For any \( s = (s_1, \ldots, s_m) \in E_{\varepsilon, m} \) and for any \( K_\varepsilon \leq k \leq m \), the maximum of \( \{s_1, \ldots, s_k\} \) is at least \( k^2 \).
4. For any \( s = (s_1, \ldots, s_m) \in E_{\varepsilon, m} \) and for any \( K_\varepsilon \leq k \leq m \), the maximum of \( \{s_1, \ldots, s_k\} \) appears in \( (s_1, \ldots, s_k) \) only once.

Proof. It is straightforward to see that \( p \) has finite entropy.

Let \( s = (s_1, s_2, \ldots) \in \mathbb{N}^\infty \) have distribution \( p^{\times \infty} \); i.e., \( s \) is chosen i.i.d. \( p \). Since each \( s_i \) has distribution \( p \), for each \( n \in \mathbb{N} \) we have:

\[
\mathbb{P}[s_i \geq n] = \sum_{m=n}^{\infty} p(m) = c \sum_{m=n}^{\infty} m^{-5/4} \geq c \int_{n}^{\infty} x^{-5/4} \, dx = 4cn^{-1/4}.
\]

For \( k \geq 1 \), let

\[ M_k := \max\{s_1, \ldots, s_k\}, \]

and let

\[ \text{next}(k) := \min\{i > k \mid s_i \geq M_k\}. \]

In words, \( \text{next}(k) \) is the first index \( i > k \) for which \( s_i \) matches or exceeds \( M_k \).

We first show that with probability one, \( M_k \geq k^2 \) for all \( k \) large enough. To this end, let \( A_k \) be the event that \( M_k < k^2 \). We have:

\[
\mathbb{P}[A_k] = \mathbb{P}[s_i < k^2 \forall i \in \{1, \ldots, k\}] = (1 - \mathbb{P}[s_1 < k^2])^k \leq (1 - 4c(k^2)^{-1/4})^k \leq e^{-4ck^{1/2}}.
\]

Since the sum of these probabilities is finite, by Borel-Cantelli we get that

\[ \mathbb{P}[A_k \text{ infinitely often}] = 0. \]
Hence $M_k \geq k^2$ for all $k$ large enough, almost surely. Furthermore, the expectation of $1/M_k$ is small:

\[(2.3)\]

$$
\mathbb{E} \left[ \frac{1}{M_k} \right] = \mathbb{E} \left[ \frac{1}{M_k} A_k \right] \mathbb{P} [A_k] + \mathbb{E} \left[ \frac{1}{M_k} \right] \mathbb{P} [-A_k] \leq e^{-4ck^{1/2}} + \frac{1}{k^2}.
$$

Next, we show that, with probability one, $s_{\text{next}(k)} > M_k$ for all $k$ large enough. That is, for large enough $k$, the first time that $M_k$ is matched or exceeded after index $k$, it is in fact exceeded.

Let $B_k$ be the event that $s_{\text{next}(k)} = M_k$. We would like to show that this occurs only finitely often. Note that

$$
\mathbb{P} [B_k | M_k] = \sum_{i=k+1}^{\infty} \mathbb{P} [s_i = M_k, \text{next}(k) = i | M_k].
$$

Applying the definition of next$(k)$ yields

$$
\mathbb{P} [B_k | M_k] = \sum_{i=k+1}^{\infty} \mathbb{P} [s_i = M_k, s_{k+1}, \ldots, s_{i-1} < M_k | M_k].
$$

By the independence of the $s_i$’s we can write this as

$$
\mathbb{P} [B_k | M_k] = \sum_{i=k+1}^{\infty} \mathbb{P} [s_i = M_k | M_k] \prod_{n=1}^{i-(k+1)} \mathbb{P} [s_{k+n} < M_k | M_k]
$$

By (2.2), $\mathbb{P} [s_{k+1} < M_k | M_k] \leq 1 - 4cM_k^{-1/4}$. Hence

$$
\mathbb{P} [B_k | M_k] \leq \frac{c}{M_k^{5/4}} \cdot \frac{1}{4cM_k^{-1/4}} = \frac{1}{4M_k}.
$$

Using (2.3) it follows that

$$
\mathbb{P} [B_k] = \mathbb{E} [\mathbb{P} [B_k | M_k]] \leq \mathbb{E} \left[ \frac{1}{4M_k} \right] \leq \frac{1}{4} e^{-4ck^{1/2}} + \frac{1}{4k^2}.
$$

Hence $\sum_k \mathbb{P} [B_k] < \infty$, and so by Borel-Cantelli $B_k$ occurs only finitely often.

Since $A_k$ and $B_k$ both occur for only finitely many $k$, the (random) index $\text{ind}'$ at which they stop occurring is almost surely finite, and is given by

$$
\text{ind}' = \min \{ \ell \in \mathbb{N} : s \not\in A_k \cup B_k \text{ for all } k \geq \ell \}.
$$
Let
\[ \text{ind} = \text{next}(\text{ind}). \]

Hence for \( k \geq \text{ind} \), \( M_k \geq k^2 \) and \( M_k \) appears in \((s_1, \ldots, s_k)\) only once.

Fix \( \varepsilon > 0 \). Since \( \text{ind} \) is almost surely finite, then for large enough constants \( K_\varepsilon \in \mathbb{N} \) and \( N_\varepsilon \in \mathbb{N} \) the event
\[ E_\varepsilon = \{ \text{ind} \leq K_\varepsilon \text{ and } M_{K_\varepsilon} \leq N_\varepsilon \} \]
has probability at least \( 1 - \varepsilon \), and additionally, conditioned on \( E_\varepsilon \) it holds that \( k \geq \text{ind} \) for all \( k \geq K_\varepsilon \), and hence \( M_k \geq k^2 \) and \( M_k \) appears in \((s_1, \ldots, s_k)\) only once. Therefore, if for \( m \geq K_\varepsilon \) we let \( E_{\varepsilon, m} \) be the projection of \( E_\varepsilon \) to the first \( m \) coordinates, then \( E_{\varepsilon, m} \) satisfies the desired properties.

\[ \square \]

2.3. Proof of Proposition 2.2. Let \( G = \{a_1, a_2, \ldots\}; \frac{1}{8} > \varepsilon > 0; p \in P(\mathbb{N}), K_\varepsilon, N_\varepsilon \in \mathbb{N}, \) and \( E_{\varepsilon, m} \subseteq \mathbb{N}^m \) be the probability measure, the constants, and the events from Lemma 2.5. To simplify notation let \( N = N_\varepsilon \) and \( K = K_\varepsilon \). Choose \( g_1, g_2, \ldots, g_N \) arbitrarily in \( G \). For any \( n \leq N \) let \( A_n = \{g_n, g_n^{-1}, a_n, a_n^{-1}\} \) and \( B_n = \cup_{i \leq n} A_i \cup \{e\} \). Choose \( h \in G \) such that \( h \notin (B_N)^{2N} \), where \((B_N)^{2N}\) is a finite set.

For \( n = N + 1, N + 2, \ldots \) we will define \( g_n \) inductively and set \( A_n = \{g_n, g_n^{-1}, a_n, a_n^{-1}\}, B_n = \cup_{i \leq n} A_i \cup \{e\} \). For any \( n \in \mathbb{N} \) let \( C_n = B_n \cup \{h^{-1}, h\} \). Note that \( A_n, B_n, \) and \( C_n \) are finite and symmetric for any \( n \in \mathbb{N} \). Now we explain how we define \( g_{N+1}, g_{N+2}, \ldots \). Let \( n + 1 > N \), and assume that \( g_1, \ldots, g_n \in G \) are defined. Let \( g_{n+1} \in G \) be a super-switching element for \((C_n)^{2n+1}\) which is not in \((C_n)^{8n+1}\). The existence of such super-switching element is guaranteed by Proposition 2.4 and the fact that \((C_n)^{2n+1}\) is a finite symmetric subset of \( G \) and \((C_n)^{8n+1}\) is finite.

For \( n \in \mathbb{N} \), define a symmetric probability measure \( \mu_n \) on \( A_n \) with
\[ \mu_n = \varepsilon 2^{-n} \left( \frac{1}{2} \delta_{a_n} + \frac{1}{2} \delta_{a_n^{-1}} \right) + (1 - \varepsilon 2^{-n}) \left( \frac{1}{2} \delta_{g_n} + \frac{1}{2} \delta_{g_n^{-1}} \right). \]

Here \( \delta_g \) is the point mass on \( g \in G \). Finally, let
\[ \mu = \sum_{n=1}^{\infty} p(n) \mu_n. \]

Obviously \( \mu \) is symmetric and has full support. Since \( p \) has finite entropy and each \( \mu_n \) has support of size at most 4, it follows easily that \( \mu \) has finite entropy.

We want to show that for \( \mu \) and \( h \) we have
\[ \lim_{m \to \infty} \| h \mu^m - \mu^m \| > 0.\]
Fix $m \in \mathbb{N}$ larger than $K$ and $N$. For each $n \in \mathbb{N}$ define $f_n : \{1, 2, 3, 4\} \to A_n$ by

$$f_n(1) = a_n, \ f_n(2) = a_n^{-1}, \ f_n(3) = g_n, \ f_n(4) = g_n^{-1},$$

and define $\nu_n : \{1, 2, 3, 4\} \to [0, 1]$ by

$$\nu_n(1) = \nu_n(2) = \frac{1}{2} \varepsilon 2^{-n}, \ \nu_n(3) = \nu_n(4) = \frac{1}{2}(1 - \varepsilon 2^{-n}).$$

Let

$$\Omega = \{(s, w) \mid s \in \mathbb{N}^m, \ w \in \{1, 2, 3, 4\}^m\}.$$ 

Let $\nu : \Omega \to [0, 1]$ be defined by

$$\nu(s, w) = p^{\chi_m(s)} \nu_1(w_1) \nu_2(w_2) \ldots \nu_m(w_m).$$

Obviously $\nu$ is a probability measure on $\Omega$.

Define $r : \Omega \to G$ by

$$r(s, w) = f_{s_1}(w_1)f_{s_2}(w_2)\ldots f_{s_m}(w_m).$$

It is not difficult to see that $r_*\nu = \mu^m$, and so we need to show that $\|hr_*\nu - r_*\nu\|$ is uniformly bounded away from zero for $m$ larger than $K$ and $N$.

Recall that $E_{\varepsilon, m} \subseteq \mathbb{N}^m$ is the event given by Lemma 2.5. Fix $s \in E_{\varepsilon, m}$. Define

$$i_{s, 1} = \min\{j \in \{1, \ldots, m\} \mid s_j > N\},$$

$$i_{s, 2} = \min\{j > i_{s, 1} \mid s_j \geq s_{i_{s, 1}}\},$$

$$\vdots$$

$$i_{s, l(s)} = \min\{j > i_{s, l(s)-1} \mid s_j \geq s_{i_{s, l(s)-1}}\}.$$ 

So, $i_{s, 1}$ is the index of the first entry of $(s_1, \ldots, s_m)$ which is larger than $N$, $i_{s, 2}$ is the index of the next entry that is at least as large, etc. Note that by the second property of $E_{\varepsilon, m}$ in Lemma 2.5, we know that

$$K < i_{s, 1} < i_{s, 2} < \cdots < i_{s, l(s)},$$

and by the fourth property,

$$N < s_{i_{s, 1}} < s_{i_{s, 2}} < \cdots < s_{i_{s, l(s)}} = \max\{s_1, \ldots, s_m\}.$$ 

Let

$$W_\varepsilon^s = \{w \in \{1, 2, 3, 4\}^m \mid \forall k \leq l(s) \ w_{i_{s, k}} = 3, 4\}.$$ 

Using the notation $\mathbb{P} \left[\cdot\right] = \nu(\cdot)$, we note that for

$$\Omega^s = \{s\} \times \{1, 2, 3, 4\}^m \subseteq \Omega$$
it holds that
\[
P \{ s \times W^s_\varepsilon | \Omega^s \} = 1 - P \{ \neg (s \times W^s_\varepsilon) | \Omega^s \}
\]
\[
= 1 - \sum_{l(s)} \mathbb{P} \left[ w_{i_{s,l}} = 1, 2; \text{ or } w_{i_{s,2}} = 1, 2; \ldots; \text{ or } w_{i_{s,l(s)}} = 1, 2 \right| \Omega^s] \\
\geq 1 - \sum_{k=1}^{l(s)} \varepsilon 2^{-s_{i_{s,k}}} \\
= 1 - \sum_{k=1}^{\infty} \varepsilon 2^{-l(s)} = 1 - \varepsilon.
\]
The last inequality is because \( s_{i_{s,1}} < s_{i_{s,2}} < \cdots < s_{i_{s,l(s)}} \).

Finally, let \( \Omega_\varepsilon = \{ (s, w) \in \Omega | s \in E_{\varepsilon,m}, w \in W^s_\varepsilon \} \), and note that
\[
\nu(\Omega_\varepsilon) \geq (1 - \varepsilon)(1 - \varepsilon) > 1 - 2\varepsilon.
\]

Claim 2.6. For any \( \alpha, \beta \in \Omega_\varepsilon \), we have \( hr(\alpha) \neq r(\beta) \).

We prove this claim after we finish the proof of the Proposition.

Let \( \nu_1 \) be equal to \( \nu \) conditioned on \( \Omega_\varepsilon \), and \( \nu_2 \) be equal to \( \nu \) conditioned on the complement of \( \Omega_\varepsilon \). We have
\[
\nu = \nu(\Omega_\varepsilon)\nu_1 + (1 - \nu(\Omega_\varepsilon))\nu_2,
\]
and by the above claim we know \( \| hr_\alpha \nu_1 - r_\alpha \nu_1 \| = 2. \) So for \( m \) larger than \( K \) and \( N \)
\[
\| h\mu^m - \mu^m \| = \| hr_\alpha \nu - r_\alpha \nu \|
\leq \nu(\Omega_\varepsilon) \| hr_\alpha \nu_1 - r_\alpha \nu_1 \| + (1 - \nu(\Omega_\varepsilon)) \| hr_\alpha \nu_2 - r_\alpha \nu_2 \|
\geq 2(1 - 2\varepsilon) = 2 - 8\varepsilon,
\]
which is uniformly bounded away from zero since \( \varepsilon < \frac{1}{8} \). Since \( \| h\mu^m - \mu^m \| \) is a decreasing sequence, this completes the proof of Proposition 2.2.

Proof of Claim 2.6. Let \( \alpha = (s, w), \beta = (t, v) \in \Omega_\varepsilon \). Hence \( \max \{ K, N \} < m, s \in E_{\varepsilon,m}, t \in E_{\varepsilon,m}, w \in W^s_\varepsilon, \) and \( v \in W^t_\varepsilon \). Assume that \( hr(\alpha) = r(\beta) \). So, we have
\[
hf_{i_1}(w_1) \cdots hf_{i_m}(w_m) = ft_1(v_1) \cdots ft_m(v_m).
\]
Let \( K < i_1 < i_2 < \cdots < i_{l(s)} \) and \( K < j_1 < j_2 < \cdots < j_{l(t)} \) be the indices we defined for \( s \) and \( t \) in the proof of Proposition 2.2. We
remind the reader that the unique maximum of \((s_1, \ldots, s_m)\) is attained at \(i_t(s)\), with a corresponding statement for \((t_1, \ldots, t_m)\) and \(j_t(t)\). So we have

\[
\begin{align*}
&\quad \frac{b_1}{b_2} h f_{s_1}(w_1) \cdots f_{s_{i_t(s)}-1}(w_{i_t(s)-1}) f_{s_{i_t(s)}}(w_{i_t(s)}) f_{s_{i_t(s)}+1}(w_{i_t(s)+1}) \cdots f_{s_m}(w_m) \\
&= f_{t_1}(v_1) \cdots f_{t_{j_t(t)-1}}(v_{j_t(t)-1}) f_{t_{j_t(t)}}(v_{j_t(t)}) f_{t_{j_t(t)+1}}(v_{j_t(t)+1}) \cdots f_{t_m}(v_m).
\end{align*}
\]

Let \(p = s_{i_t(s)} = \max\{s_1, \ldots, s_m\}\) and \(q = t_{j_t(t)} = \max\{t_1, \ldots, t_m\}\). Since \(w \in W^s\) and \(v \in W^t\), we know \(f_{s_{i_t(s)}}(w_{i_t(s)}) = p^\pm 1\) and \(f_{t_{j_t(t)}}(v_{j_t(t)}) = q^\pm 1\), so

\[
(2.4) \quad h b_1 p^{\pm 1} b_2 = c_1 p^\pm 1 c_2.
\]

Since \(p = \max\{s_1, \ldots, s_m\}\), and since \(m \geq K\), we know that \(m \leq m^2 \leq p\). So \(b_1, b_2 \in (B_0)^{p-1} \subseteq (C_0)^{p-1}\). Similarly \(c_1, c_2 \in (C_q)^{q-1}\).

Consider the case that \(p > q\). Then \(c_1, c_2; q^\pm 1 \in (C_q)^q \subseteq (C_p)^{p-1}\).

Hence \(g^\pm_1 = [b_1^{-1}]h^{-1}[c_1 p^\pm 1 c_2 b_2^{-1}]\) by (2.4), and so

\[
g_p \in (C_{p-1})^{d(p-1)} \{h, h^{-1}\}(C_{p-1})^{d(p-1)} \subseteq (C_{p-1})^{d(p-1)+1},
\]

which is a contradiction with our choice of \(g_p\), since \(p > N\). Similarly, if \(p < q\), we get a contradiction. So we can assume that \(p = q\).

If \(p = q\), then by (2.4) we have

\[
h b_1 p^{\pm 1} b_2 = c_1 p^\pm 1 c_2,
\]

and \(c_1, c_2, b_1, b_2 \in (C_{p-1})^{p-1}\). So, for \(x = c_1^{-1} h b_1 \in (C_{p-1})^{2(p-1)+1}\) we have \(g_p^{\pm 1} x g_p^{\pm 1} = c_2 b_2^{-1} \in (C_{p-1})^{2(p-1)} \subseteq (C_{p-1})^{2(p-1)+1}\). By the fact that \(g_p\) is a super-switching element for \((C_{p-1})^{2(p-1)+1}\), we get that \(x\) is the identity.

So \(h b_1 = c_1\), i.e.

\[
h f_{s_1}(w_1) \cdots f_{s_{i_t(s)}-1}(w_{i_t(s)-1}) = f_{t_1}(v_1) \cdots f_{t_{j_t(t)-1}}(v_{j_t(t)-1}).
\]

By the exact same argument, we can see this leads to a contradiction unless

\[
h f_{s_1}(w_1) \cdots f_{s_{i_t(s)}-1}(w_{i_t(s)-1}) = f_{t_1}(v_1) \cdots f_{t_{j_t(t)-1}}(v_{j_t(t)-1}).
\]

And again, this leads to a contradiction unless

\[
h f_{s_1}(w_1) \cdots f_{s_{i_t(s)}-2}(w_{i_t(s)-2}) = f_{t_1}(v_1) \cdots f_{t_{j_t(t)-2}}(v_{j_t(t)-2}).
\]

Note that if \(l(s) \neq l(t)\), at some point in this process we get that either all the \(s_i\)’s or all the \(t_i\)’s are at most \(N\) while the other string has characters strictly greater than \(N\). This leads to a contradiction.
similar to the case \( p \neq q \), which we explained before. So, by continuing this process, we get a contradiction unless

\[
h f_{s_1}(w_1) \cdots f_{s_{i_1-1}}(w_{i_1-1}) = f_{i_1}(v_1) \cdots f_{i_{j_1-1}}(v_{j_1-1}).
\]

(1) Recall that \( s_1, \ldots, s_{i_1-1} \leq N \), which means \( \max \{s_1, \ldots, s_{i_1-1}\} \leq N \).
(2) On the other hand \( s_{i_1} > N \) and we know that \( s_1, \ldots, s_K \leq N \). So, \( i_1 > K \implies i_1 - 1 \geq K \). So, by the third property of \( E_{\varepsilon,m} \) in Lemma 2.5, we get that \( \max \{s_1, \ldots, s_{i_1-1}\} \geq (i_1 - 1)^2 \geq (i_1 - 1) \).
(3) Combining the last two results we get \( N \geq i_1 - 1 \).
(4) Again, since \( s_1, \ldots, s_{i_1-1} \leq N \), we get that

\[
f_{s_1}(w_1), \ldots, f_{s_{i_1-1}}(w_{i_1-1}) \in B_N.
\]

So,

\[
b' = f_{s_1}(w_1) \cdots f_{s_{i_1-1}}(w_{i_1-1}) \in (B_N)^{i_1-1} \subseteq (B_N)^N.
\]

A similar argument shows that \( c' = f_{i_1}(v_1) \cdots f_{i_{j_1-1}}(v_{j_1-1}) \in (B_N)^N \).
But we know that \( hb' = c' \implies h = c'b'^{-1} \in (B_N)^{2N} \), which is in contradiction with our choice of \( h \).

\[\square\]

**Remark 2.7.** Note that it is possible, by a straightforward modification of this proof, to construct such a \( \mu \) for any non-identity \( h \in G \).

**References**


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