# THERE ARE NO MONOTONE HOMOMORPHISMS OUT OF THE CONVOLUTION SEMIGROUP

TOBIAS FRITZ AND OMER TAMUZ

ABSTRACT. We prove that there is no nonzero way of assigning real numbers to probability measures on $\mathbb{R}$ in a way which is monotone under first-order stochastic dominance and additive under convolution.

## 1. INTRODUCTION

With $\mathcal{P}(\mathbb{R})$ the set of probability measures on $\mathbb{R}$, it is a natural question to ask for a classification of all maps $\phi : \mathcal{P}(\mathbb{R}) \to \mathbb{R}$ such that

(a) $\phi$ is monotone with respect to first-order stochastic dominance, and
(b) $\phi$ is additive under convolution of measures.

In fact, $\mathcal{P}(\mathbb{R})$ is a partially ordered commutative monoid with respect to first-order stochastic dominance as the order relation, and convolution as the operation on measures. For partially ordered commutative monoids (and similarly for other partially ordered algebraic structures like commutative groups and rings), the set of monotone homomorphisms to $\mathbb{R}$ is a basic dual object whose study often yields fruitful insights [1, 2, 4]. The monoid $\mathcal{P}(\mathbb{R})$—considered without the partial order—is of course an important object, and the set of homomorphisms to $\mathbb{R}$ out of $\mathcal{P}(\mathbb{R})$ and various of its submonoids have been studied in the literature (see, e.g., [7, 5]).

As we will prove in this note, the only $\phi$ which satisfies both of the above conditions is $\phi = 0$ (Theorem 3). On the other hand, if one restricts to distributions that have an expectation, then taking the expectation value $\mu \mapsto \mathbb{E}\mu$ defines such a $\phi$. Thus, the inclusion of distributions with (very) fat tails is the obstruction to existence. When restricting to distributions that have all moments, it was shown in [6, Proposition 2] that the admissible $\phi$'s are precisely the scalar multiples of the expectation. Further restrictions yield additional possibilities. For example, for $\mathcal{P}(\mathbb{R}_+)$ one can take the minimum of the support. For compactly supported distributions one can take $\mu \mapsto \log \int e^{tx} d\mu(x)$ (the cumulant generating function of $\mu$) for any fixed $t > 0$.

The above question also has a motivation in terms of applied probability. Namely, we can think of $\phi(\mu)$ as a "summary statistic"—a single number that captures some important property of the distribution $\mu \in \mathcal{P}(\mathbb{R})$. Problems of this type arise in statistics, economics,

operations research and other fields. For example, in financial asset pricing, $\mu$ can describe the distribution of returns of an asset. Then what price $\phi(\mu)$ should we assign to the asset? If the mass of $\mu$ is below that of $\nu$ in the sense of first-order stochastic dominance, then we certainly expect $\phi(\mu) \leq \phi(\nu)$. While if an asset is a portfolio consisting of two other assets $\mu$ and $\nu$ assumed independent, then its return distribution is described by the convolution $\mu * \nu$, and—under mild assumptions—we would expect the prices to satisfy $\phi(\mu * \nu) = \phi(\mu) + \phi(\nu)$. A similar approach is taken in the study of risk measures (see, e.g., [3]).

## 2. Results

We consider $\mathcal{P}(\mathbb{R})$, the set of Borel probability measures on $\mathbb{R}$, as a partially ordered set with respect to *first-order stochastic dominance*, which means that $\mu \leq \nu$ if and only if their cumulative distributions functions are ordered pointwise,

$$\mu((-\infty, x]) \geq \nu((-\infty, x]) \qquad \forall x \in \mathbb{R}.$$

Equivalently, $\mu \leq \nu$ if there exists a standard probability space with random variables $X$ and $Y$ having distributions $\mu$ and $\nu$, and such that $X \leq Y$ almost surely. If $\pi \colon \mathbb{R} \to \mathbb{R}$ satisfies $\pi(x) \leq x$, then for any $\nu$ it holds that the push-forward $\pi_*\nu$ is dominated by $\nu$. Morever, for non-atomic measures, $\mu \leq \nu$ if and only if there exists some such $\pi$ with $\pi_*\nu = \mu$. Intuitively, $\mu \leq \nu$ if one can arrive at $\mu$ by starting with $\nu$ and shifting mass to the left.

We say that $\phi \colon \mathcal{P}(\mathbb{R}) \to \mathbb{R}$ is *monotone* if $\mu \leq \nu$ implies $\phi(\mu) \leq \phi(\nu)$. For $x \in \mathbb{R}$ denote by $\delta_x$ the point mass at $x$. We then say that $\phi \colon \mathcal{P}(\mathbb{R}) \to \mathbb{R}$ is *translation invariant* if $\phi(\mu * \delta_x) = \phi(\mu)$ for all $\mu \in \mathcal{P}(\mathbb{R})$ and $x \in \mathbb{R}$. Considering $\mathcal{P}(\mathbb{R})$ as a monoid under convolution and $\mathbb{R}$ as a monoid under addition, we say that $\phi$ is a *homomorphism* if $\phi(\mu * \nu) = \phi(\mu) + \mu(\nu)$.

**Lemma 1.** *Let $\phi \colon \mathcal{P}(\mathbb{R}) \to \mathbb{R}$ be translation invariant and monotone. Then $\phi$ is bounded.*

*Proof.* Let $\pi \colon \mathbb{R} \to \mathbb{R}$ be given by $\pi(x) = 0$ if $x \leq 0$ and $\pi(x) = x$ if $x > 0$. Given $\mu \in \mathcal{P}(\mathbb{R})$, denote by $\pi_*\mu$ the push-forward of $\mu$ under $\pi$. In words, to arrive at $\pi_*\mu$ we start with $\mu$, and transport the mass on the negative axis to a point mass at 0. Thus $\pi_*\mu$ is supported on $[0, \infty)$. It is immediate that $\mu \leq \pi_*\mu$.

By the translation invariance and monotonicity assumptions on $\phi$, for all $\mu \in \mathcal{P}(\mathbb{R})$ and $x \in \mathbb{R}$ it holds that $\phi(\mu * \delta_x) = \phi(\mu)$ and $\phi(\mu) \leq \phi(\pi_*\mu)$.

Assume now by contradiction that $\phi$ is unbounded from above, so that there is a sequence $(\mu_n)_{n \in \mathbb{N}}$ with $\phi(\mu_n) \geq n$. Define the sequence of measures $(\nu_n)_n$ as follows: for each $n$, choose $x$ large enough so that $\mu_n((-\infty, x]) \geq 1 - 1/n$, and let $\nu_n = \pi_*(\mu_n * \delta_{-x})$ be the translation of $\mu_n$ by $-x$, pushed-forward by $\pi$. Note that (i) $\nu_n([0, \infty)) = 1$, (ii) $\nu_n(\{0\}) \geq 1 - 1/n$, and (iii) $\phi(\nu_n) \geq \phi(\mu_n) \geq n$.

Denote by

$$F_n(x) = \nu_n((-\infty, x]) = \nu_n([0, x])$$

the cumulative distribution function of $\nu_n$, and let $F(x) = \inf_n F_n(x)$. Since each $F_n$ is right-continuous and non-decreasing it is upper-semicontinuous. Hence $F$ is also right-continuous and non-decreasing. We argue that $\lim_{x \to \infty} F(x) = 1$. This follows from the fact that for all $n \geq m$ and $x \geq 0$ it holds that $F_n(x) \geq 1 - 1/m$ (see (ii) above). By considering the finitely

many cases $F_1, \ldots, F_m$ separately, for every $m$ we can therefore find $x$ large enough so that $F_n(x) \geq 1 - 1/m$ for all $n$. Hence $\lim_{x \to \infty} \inf_n F_n(x) = 1$.

Since $F$ is right-continuous and non-decreasing, since $\lim_{x \to -\infty} F(x) = 0$, and since $\lim_{x \to \infty} F(x) = 1$, $F$ is the cumulative distribution function of some $\nu \in \mathcal{P}(\mathbb{R})$. Since $F(x) \leq F_n(x)$ for all $x$ and $n$, we have that $\nu_n \leq \nu$ for all $n$, and so $\phi(\nu) \geq n$ for all $n$. We have thus reached a contradiction.

An analogous argument with respect to going downwards in the stochastic order shows that $\phi$ must also be bounded below. $\qquad\square$

Lemma 1 is not true without the assumption of translation invariance: for example, for any $p \in (0, 1)$, taking the quantile

$$\mu \longmapsto \inf\{x \in \mathbb{R} \mid \mu((-\infty, x]) \geq p\}$$

defines an unbounded monotone map $\mathcal{P}(\mathbb{R}) \to \mathbb{R}$.

The expectation value $\mathbb{E}\mu$ either takes values in $\mathbb{R}$, or is undefined if $\mu$ does not have a first moment.

**Lemma 2.** *Let $\phi : \mathcal{P}(\mathbb{R}) \to \mathbb{R}$ be a monotone homomorphism. If $\mu$ and $\nu$ have expectations, and if $\mathbb{E}\mu < \mathbb{E}\nu$, then $\phi(\mu) \leq \phi(\nu)$.*

*Proof.* If the expectation of $\nu$ is strictly larger, then by [6, Theorem 1], there exists an $\eta \in \mathcal{P}(\mathbb{R})$ such that $\mu * \eta \leq \nu * \eta$. Hence $\phi(\mu * \eta) \leq \phi(\nu * \eta)$ by monotonicity. The conclusion now follows from additivity. $\qquad\square$

**Theorem 3.** *The only monotone homomorphism $\phi : \mathcal{P}(\mathbb{R}) \to \mathbb{R}$ is $\phi = 0$.*

*Proof.* Since $\mathcal{P}(\mathbb{R})$ has no torsion elements except its identity, $\phi$ can only be bounded if it is identically zero. Thus the claim will follow from Lemma 1 if we can show that $\phi$ is translation invariant. To this end, we need to show that for every $x \in \mathbb{R}$ it holds that $\phi(\delta_x) = 0$. This will follow if we show that $\phi(\delta_1) = 0$, since the convolution powers of $\delta_1$ eventually stochastically dominate every other point mass.

Assume then for contradiction that $\phi(\delta_1) > 0$, and normalize without loss of generality to $\phi(\delta_1) = 1$. By additivity we have that $\phi(\delta_n) = n$, and hence, by Lemma 2, for every $\mu$ with an expectation $\mathbb{E}\mu > n$ it holds that $\phi(\mu) \geq n$.

For $t \in \mathbb{R}$, let $\pi^t \colon \mathbb{R} \to \mathbb{R}$ be given by $\pi^t(x) = x$ if $x \leq t$, and $\pi^t(x) = t$ if $x > t$. Similarly to the construction in the proof of Lemma 1, $\pi_*^t \mu$ is arrived at by starting from $\mu$ and pushing all the mass in $[t, \infty)$ into a point mass at $t$. Clearly $\pi_*^t \mu \leq \mu$.

Finally, let $\mu$ be any measure that has infinite first moment and is supported on $[0, \infty)$. Then $\mu_n = \pi_*^t \mu$ has an expectation (since it has compact support), and $\lim_n \mathbb{E}\mu_n = \infty$ e.g. by monotone convergence. Hence $\lim_n \phi(\mu_n) = \infty$, since, as we note above, it follows from Lemma 2 that $\mathbb{E}\mu_n > n$ implies $\phi(\mu_n) > n$. But $\mu_n \leq \mu$, and so $\phi(\mu_n) \leq \phi(\mu)$, and we have reached the desired contradiction. $\qquad\square$

This result highlights the following open question:

**Question 4.** *Given $\mu, \nu \in \mathcal{P}(\mathbb{R})$, under what conditions is there $\eta \in \mathcal{P}(\mathbb{R})$ with $\mu * \eta \leq \nu * \eta$?*

When $\mu$ and $\nu$ are different and have finite expectation, [6] show that a necessary and sufficient condition for the strict inequality $\mu * \eta < \nu * \eta$ to hold is $\mathbb{E}\mu < \mathbb{E}\nu$.[1] If a nonzero $\phi$ as in Theorem 3 *had* existed, then $\phi(\mu) \le \phi(\nu)$ would give us a necessary condition in general. In lieu of such an obstruction existing, here is what we know in the infinite expectation cases:

(a) If $\mathbb{E}\mu = \mathbb{E}\nu = \infty$ or $\mathbb{E}\mu = E\nu = -\infty$, then it is possible as far as we know that an $\eta$ with $\mu * \eta \le \nu * \eta$ exists automatically.

(b) If $\mathbb{E}\mu = \infty$ and $\mathbb{E}\nu = -\infty$, then such an $\eta$ cannot exist, since we can easily find $\mu'$ and $\nu'$ having finite expectation and such that $\mu' \le \mu$ and $\nu \le \nu'$ while $\mathbb{E}\mu' > \mathbb{E}\nu'$. Then a putative $\mu * \eta \le \nu * \eta$ would give $\mu' * \eta \le \nu' * \eta$ as well, and hence $\mathbb{E}\mu' \le \mathbb{E}\nu'$, a contradiction.

(c) If $\mathbb{E}\mu = -\infty$ and $\mathbb{E}\nu = \infty$, then such an $\eta$ automatically exists, since we can find $\mu'$ and $\nu'$ with finite expectation such that $\mu \le \mu'$ and $\nu' \le \nu$ while $\mathbb{E}\mu' < E\nu'$, so that the result of [6] applies to $\mu'$ and $\nu'$.

(d) The remaining case is that one of the two expectations does not even converge in the extended reals, meaning that $\int_0^\infty x \, d\mu = \int_{-\infty}^0 |x| \, d\mu = \infty$ or similarly for $\nu$. We do not know what happens in this case either.

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Perimeter Institute for Theoretical Physics, Waterloo, Ontario, Canada
*Email address*: `tfritz@pitp.ca`

California Institute of Technology, Pasadena, California, USA
*Email address*: `tamuz@caltech.edu`

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[1]Sufficiency is Theorem 1 in [6]. Necessity is noted on page 18.