MONOTONE HOMOMORPHISMS ON CONVOLUTION SEMIGROUPS

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Abstract. We study monotone homomorphisms on the semigroup of probability measures on $\mathbb{R}$, by which we mean maps to the reals that are monotone with respect to the stochastic order and additive under convolution. We show that scalar multiples of the expectation are the unique monotone homomorphisms on the semigroup of measures with finite p-th moment, for any $1 \leq p < \infty$. We also prove that the entire semigroup of probability measures admits no non-zero monotone homomorphism.

1. Introduction

The set $\mathcal{P}(\mathbb{R})$ of probability measures on the reals admits a natural semigroup structure given by the operation of convolution. It also admits a natural partial order: the stochastic order, also known as stochastic dominance. In this paper we study *monotone homomorphisms*: maps from $\mathcal{P}(\mathbb{R})$, or from sub-semigroups thereof, to the reals, which are monotone with respect to the stochastic order and which are additive with respect to convolutions.

Each sub-semigroup of $\mathcal{P}(\mathbb{R})$ is a partially ordered commutative semigroup, in the sense that the stochastic order respects the semigroup operation. For partially ordered commutative semigroups (and similarly for other partially ordered algebraic structures like commutative groups and rings), the set of monotone homomorphisms to $\mathbb{R}$ is a basic dual object whose study often yields fruitful insights [1,2,4]. The monoid $\mathcal{P}(\mathbb{R})$—considered without the stochastic order—is of course an important object on its own, and the set of homomorphisms to $\mathbb{R}$ out of $\mathcal{P}(\mathbb{R})$ and various of its submonoids have been studied in the literature (see, e.g., [5,9]).

Monotone homomorphisms are also naturally motivated in terms of applied probability. Namely, we can think of a monotone homomorphism φ as a "summary statistic"—a single number $\varphi(\mu)$ that captures

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some important property of a distribution $\mu \in \mathcal{P}(\mathbb{R})$. The problem of finding well-behaved summary statistics arises in statistics, economics, operations research and other fields. For example, in financial asset pricing, μ can describe the distribution of returns of an asset. Then what price $\varphi(\mu)$ should we assign to the asset? If the mass of μ is below that of ν in the sense of first-order stochastic dominance, then we certainly expect $\varphi(\mu) \leq \varphi(\nu)$. While if an asset is a portfolio consisting of two other assets μ and ν assumed independent, then its return distribution is described by the convolution $\mu * \nu$, and—under some assumptions—we would expect the prices to satisfy $\varphi(\mu * \nu) = \varphi(\mu) + \varphi(\nu)$. A similar approach is taken in the study of risk measures (see, e.g., [3]).

Denote by $\mathscr{L}^p \subset \mathcal{P}(\mathbb{R})$ the sub-semigroup of measures with finite p-th moment, for $0 < p < \infty$. For $p \geq 1$, a monotone homomorphism $\mathscr{L}^p \to \mathbb{R}$ is given by the expectation $\mathbb{E}[\mu] = \int x \, \mathrm{d}\mu(x)$. Some natural questions are: are there other monotone homomorphisms on $\mathscr{L}^1$? On $\mathscr{L}^p$ for $p > 1$? Are there *any* monotone homomorphisms on the entire semigroup $\mathcal{P}(\mathbb{R})$ at all? We answer all of these questions: the expectation is (up to scalar multiplication) the unique monotone homomorphism on $\mathscr{L}^p$ for $1 \leq p < \infty$, and there are no non-zero monotone homomorphisms on $\mathcal{P}(\mathbb{R})$ or any $\mathscr{L}^p$ with $p < 1$. Perhaps surprisingly, we show that there are semigroups strictly between $\mathscr{L}^1$ and $\mathcal{P}(\mathbb{R})$ which do admit non-trivial monotone homomorphisms. We end the paper with a number of further examples and open questions.

1.1. **Related literature.** Ruzsa and Székely study the semigroup $\mathcal{P}(\mathbb{R})$ in their book "Algebraic Probability Theory" [9]. They devote a chapter to homomorphisms, where one of the questions they tackle is whether the expectation can be extended to all of $\mathcal{P}(\mathbb{R})$. They show in their Theorem 2.4 that there are such extensions, and moreover there are extensions that assign 0 to all symmetric distributions. However, there are no extensions that are non-negative for distributions supported on $\mathbb{R}_+$.

Mattner [5] studies the sub-semigroup $\bigcap_{p \in [1,\infty)} \mathscr{L}^p$ of measures that have all moments. He endows it with the topology of convergence in total variation, and of pointwise convergence of all moments. He shows that the cumulants are the unique continuous homomorphisms on this semigroup to $\mathbb{R}$.

The same semigroup is studied in [8], now as a partially ordered semigroup with respect to the stochastic order. It was shown there that for every $\mu, \nu \in \bigcap_p \mathscr{L}^p$ such that $\mathbb{E}[\mu] > \mathbb{E}[\nu]$, there is an $\eta \in \bigcap_p \mathscr{L}^p$ such that $\mu * \eta \geq \nu * \eta$. This result immediately implies that if φ is a monotone homomorphism and $\mathbb{E}[\mu] > \mathbb{E}[\nu]$, then $\varphi(\mu) \geq \varphi(\nu)$. It follows that scalar multiples of the expectation are the unique monotone homomorphisms on $\bigcap_p \mathscr{L}^p$.

The sub-semigroup of compactly supported measures is studied in [6]. Let us denote it by $\mathcal{P}_c(\mathbb{R})$. Then there are many monotone homomorphisms on $\mathcal{P}_c(\mathbb{R})$. Denote the normalized cumulant-generating function by

$$\hat{K}_\mu(t) = \frac{1}{t} \log \int \mathrm{e}^{tx} \, \mathrm{d}\mu(x),$$

and, as the unique continuous extensions, let $\hat{K}_\mu(0)$, $\hat{K}_\mu(-\infty)$ and $\hat{K}_\mu(\infty)$ respectively denote the expectation, the essential minimum, and the essential maximum of μ.

Any $t \in \mathbb{R} \cup \{-\infty, \infty\}$ determines a monotone homomorphism on $\mathcal{P}_c(\mathbb{R})$, given by $\mu \mapsto \hat{K}_\mu(t)$. It is shown in [6, Theorem 1] that there are essentially no more: the closed convex cone generated by these homomorphisms includes all the monotone homomorphisms. Equivalently, for each monotone homomorphism φ there is a finite Borel measure σ on $\mathbb{R} \cup \{-\infty, \infty\}$ such that

$$\varphi(\mu) = \int \hat{K}_\mu(t) \, \mathrm{d}\sigma(t) \tag{1.1}$$

for all $\mu \in \mathcal{P}_c(\mathbb{R})$.

For the set of measures μ for which $\hat{K}_\mu(t)$ is finite for all $t \in \mathbb{R}$ (i.e., measures with a moment generating function), it is shown in [6, Theorem 2] that every monotone homomorphism φ with $\varphi(\delta_1) = 1$ is likewise of the form (1.1), with the added restriction that σ is compactly supported on $\mathbb{R}$.

2. Main definitions and results

We consider $\mathcal{P}(\mathbb{R})$, the set of Borel probability measures on $\mathbb{R}$, as a partially ordered set with respect to the *stochastic order*, or *stochastic dominance*. Under this partial order, we have $\mu \leq \nu$ if and only if their cumulative distributions functions are ordered pointwise,

$$\mu((-\infty, x]) \geq \nu((-\infty, x]) \qquad \forall x \in \mathbb{R}.$$

Equivalently, $\mu \leq \nu$ if there exists a standard probability space with random variables X and Y having distributions μ and ν, and such that $X \leq Y$ almost surely. Intuitively, $\mu \leq \nu$ if one can arrive at μ by starting with ν and shifting mass to the left. If a measurable map $\pi \colon \mathbb{R} \to \mathbb{R}$ satisfies $\pi(x) \leq x$ for all x, then for any ν it holds that the push-forward $\pi_* \nu$ is dominated by ν.

Considering $\mathcal{P}(\mathbb{R})$ as a semigroup under convolution and $\mathbb{R}$ as a semigroup under addition, we say that a map $\varphi\colon \mathcal{P}(\mathbb{R}) \to \mathbb{R}$ is a *homomorphism* if $\varphi(\mu * \nu) = \varphi(\mu) + \mu(\nu)$. We say that it is *monotone* if $\mu \leq \nu$ implies $\varphi(\mu) \leq \varphi(\nu)$. For $x \in \mathbb{R}$ denote by δ_x the point mass at x. We then say that $\varphi\colon \mathcal{P}(\mathbb{R}) \to \mathbb{R}$ is *translation invariant* if $\varphi(\mu * \delta_x) = \varphi(\mu)$ for all $\mu \in \mathcal{P}(\mathbb{R})$ and $x \in \mathbb{R}$.

For $0 < p < \infty$, we denote by $\mathscr{L}^p \subset \mathcal{P}(\mathbb{R})$ the sub-semigroup of measures μ that have finite p-th moment,

$$\int |x|^p \, \mathrm{d}\mu(x) < \infty.$$

For $p \geq 1$, Minkowski's inequality shows that this is indeed a sub-semigroup. For $p < 1$ this follows from the fact that $|x+y|^p \leq |x|^p + |y|^p$.

We also consider $\mathcal{P}_{\mathrm{Cram}}(\mathbb{R})$, the semigroup of measures whose moment-generating function $\int \mathrm{e}^{tx} \, \mathrm{d}\mu(x)$ is finite for t in a neighborhood of zero which may depend on μ (Cramér's condition). This is a semigroup because of the multiplicativity of the moment-generating function under convolution. Note that it is a smaller semigroup than any $\mathscr{L}^p$, but larger than $\mathcal{P}_c(\mathbb{R})$.

Our first main result shows in particular that the expectation is the unique monotone homomorphism on these semigroups, where for $\mathscr{L}^p$ we need to assume $p \geq 1$ in order for the expectation to exist.

Theorem 1. *On the following sub-semigroups of $\mathcal{P}(\mathbb{R})$, the monotone homomorphisms φ are precisely the maps of the form $\varphi(\mu) = c\mathbb{E}[\mu]$ for some $c \geq 0$:*

(i) $\mathscr{L}^p$ for $1 \leq p < \infty$.
(ii) $\mathcal{P}_{\mathrm{Cram}}(\mathbb{R})$.

Our second main result is a non-existence result for monotone homomorphisms on two larger types of semigroups.

Theorem 2. *On the following sub-semigroups of $\mathcal{P}(\mathbb{R})$, the only monotone homomorphism φ is $\varphi = 0$:*

(i) $\mathcal{P}(\mathbb{R})$ itself.
(ii) $\mathscr{L}^p$ for $0 < p < 1$.

We also show that there are intermediate semigroups, situated between $\mathscr{L}^1$ and $\mathcal{P}(\mathbb{R})$, which do admit monotone homomorphisms (necessarily different from the expectation). Namely consider the set $\mathscr{L}^\psi \subset \mathcal{P}(\mathbb{R})$ of measures $\mu \in \mathcal{P}(\mathbb{R})$ for which the limit

$$\psi(\mu) := \lim_{n \to \infty} n \cdot \mu((n, \infty))$$

exists and is finite. Clearly this semigroup contains $\mathscr{L}^1$, since each $\mu \in \mathscr{L}^1$ satisfies $\psi(\mu) = 0$. It also includes some measures not in $\mathscr{L}^1$, such as the one with p.d.f. $h(x) = x^{-2}$ for $x \geq 1$, to which ψ assigns 1. Conversely, we have $\mathscr{L}^\psi \subset \mathscr{L}^p$ for every $p \in (0,1)$.

Theorem 3. *The set $\mathscr{L}^\psi$ is a sub-semigroup of $\mathcal{P}(\mathbb{R})$, and $\psi\colon \mathscr{L}^\psi \to \mathbb{R}$ is a monotone homomorphism.*

We also investigate a few other examples of intermediate semigroups, one in which the set of monotone homomorphisms is an infinite-dimensional vector space corresponding to the solutions of the Cauchy functional equation.

3. Proof of Theorem 1: Uniqueness of the expectation

Suppose μ and ν are not ranked in the stochastic order. Is it possible that there exists a measure η such that $\mu * \eta \geq \nu * \eta$? This question was first asked, independently, by Tarsney [10] and Pomatto et al. [8]. The latter show that this happens whenever $\mathbb{E}[\mu] > \mathbb{E}[\nu]$, using a lemma due to Ruzsa and Székely [9].

In general, it is impossible to control η without controlling μ and ν. In particular, even when $\mu, \nu \in \mathscr{L}^1$, there may not exist such an η in $\mathscr{L}^1$. The next lemma, which is a key ingredient of the proof of Theorem 1, shows that when μ, ν are in $\mathcal{P}_{\mathrm{Cram}}(\mathbb{R})$, one can take η to have Laplace distribution, and in particular to also be in $\mathcal{P}_{\mathrm{Cram}}(\mathbb{R})$. This was shown for compactly supported μ, ν in [7, Theorem 1] and [8, Theorem 3].

For $r > 0$, let the *Laplace measure with parameter* r be the probability measure on the real line having density function $h(x) = \frac{r}{2}\mathrm{e}^{-r|x|}$.

Lemma 3.1. *Suppose that $\mu, \nu \in \mathcal{P}_{\mathrm{Cram}}(\mathbb{R})$ satisfy $\mathbb{E}[\mu] > \mathbb{E}[\nu]$. Let η_r be the Laplace measure with parameter r. Then*

$$\mu * \eta_r \geq \nu * \eta_r$$

for every small enough $r > 0$.

Proof. Denote the c.d.f.s of μ and ν by $F(x) = \mu((-\infty, x])$ and $G(x) = \nu((-\infty, x])$ respectively.

Since $\mu, \nu \in \mathcal{P}_{\mathrm{Cram}}(\mathbb{R})$, there is some $s > 0$ such that the moment generating functions M_μ and M_ν are finite on $[-s, s]$. It follows from integration by parts that for $r \in [-s, s]$,

$$M_\mu(r) - M_\nu(r) = r \int_{-\infty}^{\infty} \mathrm{e}^{rx}[G(x) - F(x)]\,\mathrm{d}x.$$

Hence
$$\int_{-\infty}^{\infty} \mathrm{e}^{s|x|}[G(x) - F(x)]\,\mathrm{d}x \tag{3.1}$$
is well defined and finite.

For $r \in [0, s]$ and $y \in \mathbb{R}$, let $e_{y,r} \colon \mathbb{R} \to \mathbb{R}$ be given by
$$e_{y,r}(x) = \mathrm{e}^{r|y|}\mathrm{e}^{-r|y-x|}.$$
We make two observations:

(1) $e_{y,r}(x) \leq \mathrm{e}^{s|x|}$ for any $r \in [0, s]$ and $y \in \mathbb{R}$.
(2) Suppose that $(r_n)_n$ is a sequence in $[0, s]$ that converges to 0, and $(y_n)_n$ is any sequence. Then the sequence of functions $(e_{y_n,r_n})_n$ converges pointwise to 1.

Given these observations, it follows from dominated convergence[1] that for any sequences $r_n \to 0$ and y_n,
$$\begin{aligned}
&\lim_n \int_{-\infty}^{\infty} e_{y_n,r_n}(x)[G(x) - F(x)]\,\mathrm{d}x \\
&= \int_{-\infty}^{\infty} [G(x) - F(x)]\,\mathrm{d}x = \mathbb{E}[\mu] - \mathbb{E}[\nu] > 0.
\end{aligned} \tag{3.2}$$

We claim that for all r small enough it holds that
$$\Delta_r(y) := \int_{-\infty}^{\infty} \mathrm{e}^{-r|y-x|}[G(x) - F(x)]\,\mathrm{d}x$$
is positive for all y. If not, then for any sequence r_n tending to zero we can choose a sequence y_n so that $\Delta_{r_n}(y_n) \leq 0$ for all n. It follows that $\mathrm{e}^{r|y|}\Delta_{r_n}(y_n) \leq 0$, which contradicts (3.2).

Fix some such r small enough, and let $h(x) = \frac{r}{2}\mathrm{e}^{-r|x|}$ be the p.d.f. of the Laplace measure η with parameter r. The c.d.f. of $\mu * \eta$ is $h * F$, and the c.d.f. of $\nu * \eta$ is $h * G$. Furthermore
$$h * G - h * F = \frac{r}{2}\Delta_r > 0,$$
and so $\mu * \eta \geq \nu * \eta$. □

Lemma 3.2. *Let $\mathcal{S} \subset \mathcal{P}(\mathbb{R})$ be a sub-semigroup that contains the point masses $\{\delta_c : c \in \mathbb{R}\}$, and let $\varphi \colon \mathcal{S} \to \mathbb{R}$ be a monotone homomorphism. Then $\varphi(\delta_c) = c\varphi(1)$.*

Proof. By additivity, $\varphi(\delta_c) = c\varphi(1)$ for rational c. By monotonicity it follows that this holds for all c. □

[1] In more detail, the signs of the integrands in the sequence are the same for every x, and given by the sign of $G(x) - F(x)$. One can therefore apply dominated convergence to the positive and negative parts separately, with the corresponding parts of (3.1) as the dominating integral.

Lemma 3.3. *Let $\varphi\colon \mathscr{L}^p \to \mathbb{R}$ be a monotone homomorphism, and let $\mu, \nu \in \mathcal{P}_c(\mathbb{R})$ have the same expectation. Then for every $\zeta \in \mathscr{L}^p$ and $\alpha \in [0,1]$, it holds that*

$$\varphi(\alpha\mu + (1-\alpha)\zeta) = \varphi(\alpha\nu + (1-\alpha)\zeta).$$

Proof. It suffices to prove the inequality $\geq$. Fix some $c > 0$. Then $\mathbb{E}[\mu * \delta_c] > \mathbb{E}[\nu]$, and so by Lemma 3.1 there exists a Laplace measure $\eta \in \mathscr{L}^p$ such that $(\mu * \delta_c) * \eta \geq \nu * \eta$. Clearly we also have $(\zeta * \delta_c) * \eta \geq \zeta * \eta$. So

$$\alpha(\mu * \delta_c * \eta) + (1-\alpha)(\zeta * \delta_c * \eta) \geq \alpha(\nu * \eta) + (1-\alpha)(\zeta * \eta).$$

Hence

$$\big[\alpha\mu + (1-\alpha)\zeta\big] * \delta_c * \eta \geq \big[\alpha\nu + (1-\alpha)\zeta\big] * \eta.$$

It now follows by the monotonicity and additivity of φ that

$$\varphi\big(\alpha\mu + (1-\alpha)\zeta\big) + \varphi(\delta_c) \geq \varphi\big(\alpha\nu + (1-\alpha)\zeta\big).$$

By Lemma 3.2 we have $\varphi(\delta_c) = c\varphi(\delta_1)$, and so letting $c \to 0$ yields the desired inequality. ☐

By Lemma 3.1, part (ii) of Theorem 1 is a consequence of the following observation.

Proposition 3.4. *Let $\mathcal{S} \subseteq \mathscr{L}^1$ be any sub-semigroup such that:*

(1) $\mathcal{S}$ contains the point masses $\{\delta_c : c \in \mathbb{R}\}$.
(2) If $\mu, \nu \in \mathcal{S}$ satisfy $\mathbb{E}[\mu] > \mathbb{E}[\nu]$, then there is $\eta \in \mathcal{S}$ with $\mu * \eta \geq \nu * \eta$.

Then the monotone homomorphisms on $\mathcal{S}$ are precisely the scalar multiples of $\mathbb{E}$.

Proof. Given a monotone homomorphism $\varphi\colon \mathcal{S} \to \mathbb{R}$, we show that $\varphi(\mu) = \mathbb{E}[\mu]\,\varphi(\delta_1)$ for every $\mu \in \mathcal{S}$.

Fix arbitary $\varepsilon > 0$. Then by assumption we have $\eta_\pm \in \mathcal{S}$ such that

$$\delta_{\mathbb{E}[\mu]-\varepsilon} * \eta_- \leq \mu * \eta_-, \qquad \mu * \eta_+ \leq \delta_{\mathbb{E}[\mu]+\varepsilon} * \eta_+,$$

so that applying φ results in

$$\varphi(\delta_{\mathbb{E}[\mu]-\varepsilon}) \leq \varphi(\mu) \leq \varphi(\delta_{\mathbb{E}[\mu]+\varepsilon}),$$

or equivalently by Lemma 3.2,

$$(\mathbb{E}[\mu] - \varepsilon)\varphi(\delta_1) \leq \varphi(\mu) \leq (\mathbb{E}[\mu] + \varepsilon)\varphi(\delta_1).$$

This proves the claim in the limit $\varepsilon \to 0$. ☐

We now focus on the more involved proof of part (i) of Theorem 1. Let $\varphi\colon \mathscr{L}^p \to \mathbb{R}$ be a monotone homomorphism. Write $\lambda = \varphi(\delta_1)$. By monotonicity and since $\varphi(0) = 0$, we have $\lambda \geq 0$. From the above Lemma 3.3 (with $\alpha = 1$), we deduce that for any compactly supported μ,

$$\varphi(\mu) = \varphi(\delta_{\mathbb{E}[\mu]}) = \lambda \mathbb{E}[\mu].$$

We next show that the same holds for any $\mu \in \mathscr{L}^p$ that is *bounded from below* (i.e., $\mu([-M,\infty)) = 1$ for some M) but not bounded from above. Suppose for the sake of contradiction that there exists such a μ with $\varphi(\mu) \neq \lambda\mathbb{E}[\mu]$. Shifting μ by a constant if necessary, we can assume that μ is supported on $\mathbb{R}_+$, and $\varphi(\mu) \neq \lambda\mathbb{E}[\mu]$. For each positive integer n, consider the function f^n with values

$$f^n(x) := \min\{x, n\}. \tag{3.3}$$

Consider also the compactly supported measure μ_n given by the push-forward $f^n_*\mu$. That is, μ_n is obtained by shifting all the mass in μ that is to the right of n to an atom at n. Note that $\mu \geq \mu_n$.

It follows from the Monotone Convergence Theorem that $\mathbb{E}[\mu_n] \to \mathbb{E}[\mu]$. By monotonicity, we have $\varphi(\mu) \geq \varphi(\mu_n) = \lambda\mathbb{E}[\mu_n]$ for each n. Hence $\varphi(\mu) \geq \lambda\mathbb{E}[\mu]$. Since by assumption equality does not hold, we deduce $\varphi(\mu) > \lambda\mathbb{E}[\mu]$. We can then choose a large k such that $\varphi(\mu) \geq \lambda\mathbb{E}[\mu] + \frac{1}{k}$. Let $\nu_n = \mu^{(nk)}$ equal the convolution of μ with itself nk times. Then additivity of φ and $\mathbb{E}$ implies

$$\varphi(\nu_n) \geq \lambda\mathbb{E}[\nu_n] + n.$$

For each n, we now choose a sufficiently large positive number a_n that satisfies the following properties:

(1) $p_n := \nu_n([0, a_n]) \geq 1 - \frac{1}{n}$;
(2) $\int_{a_n}^{\infty} x^p \, \mathrm{d}\nu_n(x) \leq 2^{-n}$.

Denote by ν_n^1 and ν_n^2 the measure ν_n, conditioned on $[0, a_n]$ and (a_n, ∞), respectively:

$$\nu_n^1(A) = \frac{\nu_n(A \cap [0, a_n])}{p_n},$$
$$\nu_n^2(A) = \frac{\nu_n(A \cap (a_n, \infty))}{1 - p_n}.$$

Then

$$\nu_n = p_n\nu_n^1 + (1 - p_n)\nu_n^2.$$

Denote $m_n = \mathbb{E}[\nu_n^1]$, and let

$$\zeta_n = p_n\delta_{m_n} + (1-p_n)\nu_n^2.$$

That is, ζ_n is obtained from ν by contracting all the mass below a_n to an atom at its (conditional) expectation $m_n = \mathbb{E}[\nu_n^1]$. It thus follows from Lemma 3.3 that

$$\varphi(\zeta_n) = \varphi(\nu_n).$$

Next, let $\eta_n := \zeta_n * \delta_{-m_n}$ be the measure ζ_n shifted to the left by the conditional expectation; note that it is supported on $\mathbb{R}_+$ and is unbounded from above. By additivity,

$$\begin{aligned} \varphi(\eta_n) &= \varphi(\zeta_n) - \lambda\mathbb{E}[\nu_n | x \le a_n] \\ &\ge \varphi(\nu_n) - \lambda\mathbb{E}[\nu_n] \\ &\ge n, \end{aligned}$$

so that $\varphi(\eta_n)$ is large. On the other hand, its p-th moment is small: since

$$\eta_n = p_n\delta_0 + (1-p_n)\nu_n^2 * \delta_{-m_n},$$

it follows from the second property of a_n that

$$\int_0^\infty x^p \, \mathrm{d}\eta_n(x) = \int_{a_n}^\infty (x-m_n)^p \, \mathrm{d}\nu_n(x) < \int_{a_n}^\infty x^p \, \mathrm{d}\nu_n(x) \le 2^{-n}.$$

Moreover, the first condition on a_n ensures that η_n has a mass point of size at least $1 - \frac{1}{n}$ at zero.

Now define F_n to be the c.d.f. of η_n, and consider $F(x) = \inf_n F_n(x)$. Clearly $F(x) = 0$ for $x < 0$, and for every $x \ge 0$ the infimum is also achieved at some n, since when n is large $F_n(x) \ge F_n(0) \ge 1 - \frac{1}{n} > F_1(x)$. Because each $F_n(x)$ is non-decreasing and right-continuous, so is the function $F(x)$. Moreover, the fact that $F_n(0) \ge 1 - \frac{1}{n}$ and each $F_n(x) \to 1$ as $x \to \infty$ implies that $F(x) \to 1$ as $x \to \infty$. Hence F is the c.d.f. of some probability measure η supported on $\mathbb{R}_+$. In addition,

$\eta \in \mathscr{L}^p$ because

$$\begin{aligned}\int x^p \,\mathrm{d}\eta(x) &= \int_0^\infty p x^{p-1}(1-F(x))\,\mathrm{d}x \\ &\le \int_0^\infty p x^{p-1}\left(\sum_{n=1}^\infty (1-F_n(x))\right)\mathrm{d}x \\ &= \sum_{n=1}^\infty \int_0^\infty p x^{p-1}(1-F_n(x))\,\mathrm{d}x \\ &= \sum_{n=1}^\infty \int_0^\infty x^p \,\mathrm{d}\eta_n(x) \\ &\le \sum_{n=1}^\infty 2^{-n},\end{aligned}$$

where we have used the fact that for each x there exists n such that $1-F(x) = 1-F_n(x)$, and the Monotone Convergence Theorem.

Now observe that since $F(x) \le F_n(x)$ for all x, we have $\eta \ge \eta_n$. Therefore $\varphi(\eta) \ge \varphi(\eta_n) \ge n$ for each n. This contradicts the assumption that $\varphi(\eta) \in \mathbb{R}$.

Therefore, for any μ that is bounded from below, we must have $\varphi(\mu) = \lambda \mathbb{E}[\mu]$. A symmetric argument shows the same is true for any μ that is bounded from above. Finally, for a general μ that may be unbounded on both sides, we have by monotonicity, with $f^n(x) = \min\{x, n\}$ from (3.3),

$$\varphi(\mu) \ge \varphi(f^n_*\mu) = \lambda \mathbb{E}[\varphi(f^n_*\mu)],$$

so that $\varphi(\mu) \ge \lambda \mathbb{E}[\mu]$ by letting $n \to \infty$. Likewise, if we denote $g^n(x) = \max\{x, -n\}$, then

$$\varphi(\mu) \le \varphi(g^n_*\mu) = \lambda \mathbb{E}[g^n_*\mu],$$

so that $\varphi(\mu) \le \lambda \mathbb{E}[\mu]$ also holds. This concludes the proof of Theorem 1.

As a final note, in [8] it was shown that the only monotone homomorphisms on $\bigcap_{p=1}^\infty \mathscr{L}^p$ are again the scalar multiples of expectation. The argument in the above proof can be adapted to reproduce that result.

For this we just need to choose a_n sufficiently large so that

$$\int_{a_n}^\infty x^p \,\mathrm{d}\nu_n(x) \le 2^{-n} \text{ for each } p \in \{1, \dots, n\}.$$

Then $\int_0^\infty x^p \,\mathrm{d}\eta_n(x) \leq 2^{-n}$ for $p \leq n$ and for any p,

$$\int x^p \,\mathrm{d}\eta(x) \leq \sum_{n=1}^{\infty} \int x^p \,\mathrm{d}\eta_n(x) < \infty,$$

so that $\eta \in \bigcap_{p=1}^{\infty} \mathscr{L}^p$ as well.

4. Proof of Theorem 2

In this subsection we prove Theorem 2 on the non-existence of monotone homomorphisms on $\mathcal{P}(\mathbb{R})$ and $\mathscr{L}^p$ for $0 < p < 1$. We treat the case of $\mathcal{P}(\mathbb{R})$ itself first, again starting with a lemma.

As already mentioned, we say that $\varphi\colon \mathcal{P}(\mathbb{R}) \to \mathbb{R}$ is *translation invariant* if $\varphi(\mu * \delta_c) = \varphi(\mu)$ for all $\mu \in \mathcal{P}(\mathbb{R})$ and $c \in \mathbb{R}$.

Lemma 4.1. *Let $\varphi : \mathcal{P}(\mathbb{R}) \to \mathbb{R}$ be translation invariant and monotone. Then φ is bounded.*

Proof. Let $g\colon \mathbb{R} \to \mathbb{R}$ be given by $g(x) = \max\{0, x\}$. Given $\mu \in \mathcal{P}(\mathbb{R})$, denote by $g_*\mu$ the push-forward of μ under g. In words, to arrive at $g_*\mu$ we start with μ, and transport the mass on the negative axis to a point mass at 0. Thus $g_*\mu$ is supported on $[0, \infty)$. It is immediate that $\mu \leq g_*\mu$.

By the translation invariance and monotonicity assumptions on φ, for all $\mu \in \mathcal{P}(\mathbb{R})$ and $x \in \mathbb{R}$ it holds that $\varphi(\mu * \delta_x) = \varphi(\mu)$ and $\varphi(\mu) \leq \varphi(g_*\mu)$.

Assume now by contradiction that φ is unbounded from above, so that there is a sequence $(\mu_n)_{n\in\mathbb{N}}$ with $\varphi(\mu_n) \geq n$. Define the sequence of measures $(\nu_n)_n$ as follows: for each n, choose x large enough so that $\mu_n((-\infty, x]) \geq 1 - 1/n$, and let $\nu_n = g_*(\mu_n * \delta_{-x})$ be the translation of μ_n by $-x$, pushed-forward by g. Note that (i) $\nu_n([0, \infty)) = 1$, (ii) $\nu_n(\{0\}) \geq 1 - 1/n$, and (iii) $\varphi(\nu_n) \geq \varphi(\mu_n) \geq n$.

Denote by

$$F_n(x) = \nu_n((-\infty, x]) = \nu_n([0, x])$$

the c.d.f. of ν_n, and let $F(x) = \inf_n F_n(x)$. As in the proof of Theorem 1, F is non-decreasing, right-continuous, and satisfies $F(x) \to 1$ as $x \to \infty$. Thus F is the cumulative distribution function of some $\nu \in \mathcal{P}(\mathbb{R})$. Since $F(x) \leq F_n(x)$ for all x and n, we have that $\nu \geq \nu_n$ for all n, and so $\varphi(\nu) \geq \varphi(\nu_n) \geq n$ for all n. We have thus reached a contradiction.

An analogous argument with respect to going downwards in the stochastic order shows that φ must also be bounded below. $\square$

Lemma 4.1 does not require φ to be a homomorphism, and is clearly not true without the assumption of translation invariance: for example, for any $p \in (0,1)$, taking the quantile

$$\mu \longmapsto \inf\{x \in \mathbb{R} \mid \mu((-\infty, x]) \geq p\}$$

defines an unbounded monotone map $\mathcal{P}(\mathbb{R}) \to \mathbb{R}$.

We now return to the proof of Theorem 2(i). Since $\mathbb{R}$ has no nonzero torsion elements, φ can only be bounded if it is identically zero. Thus the claim will follow from Lemma 4.1 if we can show that every monotone homomorphism $\varphi \colon \mathcal{P}(\mathbb{R}) \to \mathbb{R}$ is translation invariant. To this end, we need to show that for every $x \in \mathbb{R}$ it holds that $\varphi(\delta_x) = 0$. By Lemma 3.2 it suffices to show $\varphi(\delta_1) = 0$.

Assume then for contradiction that $\varphi(\delta_1) > 0$, and normalize without loss of generality to $\varphi(\delta_1) = 1$. By additivity we have that $\varphi(\delta_n) = n$. Since the restriction of φ to $\mathscr{L}^1$ is a monontone homomorphism on $\mathscr{L}^1$, it follows from Theorem 1 that $\varphi(\mu) = \mathbb{E}[\mu]$ for every $\mu \in \mathscr{L}^1$ with an expectation.

As in the proof of Theorem 1, let $f^n(x) = \min\{x, n\}$, so that $f^n_* \mu$ is obtained by starting from μ and pushing all the mass in $[n, \infty)$ into a point mass at n. Clearly $f^n_* \mu \leq \mu$.

Finally, let μ be any measure that has infinite expectation and is supported on $[0, \infty)$. Then $\mu_n = f^n_* \mu$ has an expectation (since it has compact support), and $\lim_n \mathbb{E}[\mu_n] = \infty$ by monotone convergence. Hence $\lim_n \varphi(\mu_n) = \infty$. But $\mu \geq \mu_n$, and so $\varphi(\mu) \geq \varphi(\mu_n)$. This gives a contradiction and concludes the proof of part (i) of Theorem 2.

Part (ii) claims that there is likewise no monontone homomorphism on $\mathscr{L}^p$ for $0 < p < 1$, which we prove now. Such $\mathscr{L}^p$ contains a measure μ supported on $\mathbb{R}_+$ with infinite expectation. Approximating μ from below by $f^n_* \mu$, we deduce by monotonicity and the case of $\mathscr{L}^1$ from Theorem 1 that $\varphi(\mu) \geq \varphi(\delta_1)\, \mathbb{E}[f^n_* \mu]$ for all n. Since $\mathbb{E}[f^n_* \mu] \to \infty$ as $n \to \infty$, we obtain $\varphi(\delta_1) = 0$, and therefore φ is translation invariant as above.

The rest of the argument proceeds similarly to the proof of Theorem 1: if $\varphi(\mu) > 0$ for some positively supported $\mu \in \mathscr{L}^p$, then we can find a sequence $\nu_n \in \mathscr{L}^p$ such that $\varphi(\nu_n) \geq n$. By choosing a_n sufficiently large, we can make the resulting measures η_n satisfy $\eta_n(\{0\}) \geq 1 - \frac{1}{n}$ and $\int x^p \, \mathrm{d}\eta_n(x) \leq 2^{-n}$. Moreover, $\varphi(\eta_n) = \varphi(\nu_n) \geq n$ by translation invariance. Hence the stochastically dominant η satisfies $\varphi(\eta) \geq \varphi(\eta_n) \geq n$ for each n, leading to a contradiction. Therefore $\varphi(\mu) = 0$ for any positively supported μ, and thus for any $\mu \in \mathscr{L}^p$ that is bounded from below. Symmetrically $\varphi(\mu) = 0$ for any μ bounded from above. The general case follows from the same approximation argument as at the end of the proof of Theorem 1.

5. Other intermediate semigroups

In this section we prove Theorem 3, and exemplify the diverse behavior of the monotone homomorphisms on various other sub-semigroups of $\mathcal{P}(\mathbb{R})$.

5.1. **Existence of monotone homomorphisms.** We start with the proof of Theorem 3.

The first claim is that $\mathscr{L}^\psi$ is also a semigroup; we do not prove this separately, since it is a direct consequence of the following additivity argument for ψ. Assuming that $\mu, \nu \in L^\psi$ and $\eta = \mu * \nu$, we show that $\psi(\eta) = \psi(\mu) + \psi(\nu)$. We do this by estimating $n \cdot \eta((n, \infty))$ from both sides.

For this proof it will be useful to use probabilistic notation. Let X and Y be two independent random variables, with distributions μ and ν. Hence their sum $X + Y$ has distribution η.

Fix any $\epsilon > 0$, and let M be sufficiently large such that $\mathbb{P}[|X| \leq M]$ and $\mathbb{P}[|Y| \leq M]$ are both larger than $1 - \epsilon$. Then for every $n \geq 0$,

$$\begin{aligned}
\eta((n,\infty)) &= \mathbb{P}[X + Y > n] \\
&\geq \mathbb{P}[|X| \leq M,\ Y > n + M] + \mathbb{P}[|Y| \leq M,\ X > n + M] \\
&= \mathbb{P}[|X| \leq M] \cdot \mathbb{P}[Y > n + M] + \mathbb{P}[|Y| \leq M] \cdot \mathbb{P}[X > n + M] \\
&\geq (1 - \epsilon)\left(\mathbb{P}[Y > n + M] + \mathbb{P}[X > n + M]\right).
\end{aligned}$$

Thus

$$\begin{aligned}
&n \cdot \eta((n,\infty)) \\
&\geq \frac{(1-\epsilon)n}{n + M} \cdot ((n + M)\mathbb{P}[Y > n + M] + (n + M)\mathbb{P}[X > n + M])
\end{aligned}$$

Letting $n \to \infty$ thus yields

$$\liminf_{n\to\infty} n \cdot \eta((n,\infty)) \geq (1-\epsilon)(\psi(\mu) + \psi(\nu)).$$

Since ϵ is arbitrary, we have $\liminf_{n\to\infty} n \cdot \eta((n,\infty)) \geq \psi(\mu) + \psi(\nu)$.

In the opposite direction, for $n > 2M$ we can write as the following sum

$$\begin{aligned}
\mathbb{P}[X + Y > n] = \mathbb{P}&[X \leq M,\ X + Y > n] \\
&+ \mathbb{P}[Y \leq M,\ X + Y > n] \\
&+ \mathbb{P}[X > M, Y > M, X + Y > n].
\end{aligned}$$

These three terms can be separately bounded from *above* by $\mathbb{P}[Y > n - M]$, $\mathbb{P}[X > n - M]$ and $\mathbb{P}[X > M] \cdot \mathbb{P}[Y > n/2] + \mathbb{P}[Y > M] \cdot \mathbb{P}[X > n/2]$, respectively. Observe that

$$\lim_{n\to\infty} n \cdot \mathbb{P}[Y > n - M] = \psi(\nu),$$
$$\lim_{n\to\infty} n \cdot \mathbb{P}[X > n - M] = \psi(\mu),$$
$$\lim_{n\to\infty} n \cdot \mathbb{P}[X > M] \cdot \mathbb{P}[Y > n/2] \leq 2\epsilon \cdot \psi(\nu),$$
$$\lim_{n\to\infty} n \cdot \mathbb{P}[Y > M] \cdot \mathbb{P}[X > n/2] \leq 2\epsilon \cdot \psi(\mu).$$

Thus we obtain

$$\limsup_{n\to\infty} n \cdot \eta((n,\infty)) \leq (1 + 2\epsilon)(\psi(\mu) + \psi(\nu)).$$

Letting $\epsilon \to 0$ then yields $\limsup_{n\to\infty} n \cdot \eta((n,\infty)) \leq \psi(\mu) + \psi(\nu)$. So $\psi(\eta) = \lim_{n\to\infty} n \cdot \eta((n,\infty))$ exists and equals the sum $\psi(\mu) + \psi(\nu)$. This shows ψ is additive. It is immediate that ψ is monotone, which concludes the proof of Theorem 3.

5.2. **Further examples.** Given a measure μ, denote by $\hat{\mu}$ the reflection of μ around 0, given by $\hat{\mu}(A) := \mu(-A)$. Let $\mathscr{L}^{\psi}_{\pm}$ be the semigroup of measures μ for which both $\psi(\mu)$ and $\psi(\hat{\mu})$ exist and are finite. This space lies strictly between $\mathscr{L}^1$ and $\mathscr{L}^p$ for any $p < 1$. Despite the fact that zero is the only monotone homomorphism on the latter $\mathscr{L}^p$ semigroups, for any $a, b \geq 0$ the assignment

$$\mu \longmapsto a\psi(\mu) - b\psi(\hat{\mu})$$

defines a monotone homomorphism on the space $\mathscr{L}^{\psi}_{\pm}$. We do not know if these are the only monotone homomorphisms on this space.

We can obtain other interesting examples by considering even smaller semigroups that still contain $\mathscr{L}^1$. Let $\mathscr{L}^{\psi}_{=}$ be the set of $\mu \in L^{\psi}$ for which $\psi(\mu) = \psi(\hat{\mu})$. This space lies strictly between $\mathscr{L}^1$ and $\mathscr{L}^{\psi}_{\pm}$. Moreover, since $\mu \geq \nu$ only if $\psi(\mu) \geq \psi(\nu)$ and $\psi(\hat{\mu}) \leq \psi(\hat{\mu})$, stochastic dominance on $\mathscr{L}^{\psi}_{=}$ requires $\psi(\mu) = \psi(\hat{\mu}) = \psi(\nu) = \psi(\hat{\nu})$. Now take $p : \mathbb{R}_+ \to \mathbb{R}$ to be any additive function, i.e., any function that satisfies the Cauchy functional equation $p(x + y) = p(x) + p(y)$. Then $p \circ \psi$ is a homomorphism on $\mathscr{L}^{\psi}_{=}$, and is also trivially monotone. Thus the set of monotone homomorphisms on the space $\mathscr{L}^{\psi}_{=}$ is rather complex.

We can consider the smaller space $\mathscr{L}^{\psi}_0 \subset \mathscr{L}^{\psi}$ of measures μ for which $\psi(\mu) = \psi(\hat{\mu}) = 0$. We claim that zero is the only monotone homomorphism on this space; the proof is almost identical to the proof of Theorem 1, except that the a_n are chosen large enough so that $x \cdot \nu_n((x,\infty)) \leq \frac{1}{n}$ for all $x \geq a_n$.

For a final example, denote

$$\psi'(\mu) = \lim_{n\to\infty} n \log n \cdot \mu((n,\infty))$$

for any μ for which this limit exists. Fix a *particular* measure ζ such that $\psi(\zeta) = \psi(\hat{\zeta}) = 0$, but $\psi'(\zeta) = \psi'(\hat{\zeta}) = 1$. Consider the semigroup generated by $\mathscr{L}^1$ and ζ, which we denote by $\mathscr{L}^1_\zeta$. Choose an arbitrary $a \in \mathbb{R}$, and define

$$\varphi(\mu * \zeta^{(n)}) := \mathbb{E}[\mu] + n \cdot a$$

for all $n \in \{0, 1, \ldots\}$ and $\mu \in \mathscr{L}^1$. Then φ is clearly a homomorphism; note that well-definedness is not clear at this stage, but will follow from the following monotonicity argument.

To show that φ is monotone, note first that since $\sum_n \frac{1}{n \log n}$ diverges, for each $\mu \in \mathscr{L}^1$ we have $\psi'(\mu) = \psi'(\hat{\mu}) = 0$. Together with the assumption on ζ, we can show (using an analogous argument to the above proof that ψ is additive) that, for any k,

$$\psi'(\mu * \zeta^{(k)}) = k, \qquad \psi'(\hat{\mu} * \hat{\zeta}^{(k)}) = k.$$

From this it follows that $\mu * \zeta^{(k)}$ stochastically dominates $\nu * \zeta^{(m)}$ only if $k = m$. Moreover, in case that $k = m$ holds, if $\mu * \zeta^{(k)} \geq \nu * \zeta^{(k)}$, then it must be that $\mathbb{E}[\mu] \geq \mathbb{E}[\nu]$; see the comment at the end of Appendix A in [8]. This proves that φ is monotone, and at the same time that it is well-defined.

The significance of this last example is that although the space $\mathscr{L}^1_\zeta$ strictly contains $\mathscr{L}^1$, the monotone homomorphism φ is non-trivial when restricted to the sub-semigroup $\mathscr{L}^1$. By a now familiar approximation argument, this cannot happen on any super-semigroup of $\mathscr{L}^1$ that contains a measure μ such that exactly one of the two integrals $\int_{-\infty}^0 |x| \,\mathrm{d}\mu(x)$ and $\int_0^\infty |x| \,\mathrm{d}\mu(x)$ is infinite. For example, this is the case for the space $\mathscr{L}^\psi_=$ (or any larger space), as we have seen. In contrast, the space $\mathscr{L}^1_\zeta$ is constructed in such a way that for any $\mu \in \mathscr{L}^1_\zeta$, either $\mu \in \mathscr{L}^1$ or both integrals above diverge. In the latter case the approximation argument does not apply, which explains why φ can be non-zero on $\mathscr{L}^1$.

5.3. **Overview of all semigroups considered.** To summarize, the (strict) inclusion relationships among the semigroups discussed so far is as follows. Here q is assumed to be in $(1, \infty)$, and $p \in (0, 1)$.

$$\begin{aligned}\mathcal{P}_{\mathrm{Cram}}(\mathbb{R}) \subset \bigcap_{q>1} \mathscr{L}^q \subset & \mathscr{L}^1 \\ & \subset \mathscr{L}^1_\zeta \subset \mathscr{L}^\psi_0 \subset \mathscr{L}^\psi_= \subset \mathscr{L}^\psi_\pm \subset \mathscr{L}^\psi \subset \mathscr{L}^p \subset \mathcal{P}(\mathbb{R}),\end{aligned}$$

Recall that $\mathcal{P}_{\mathrm{Cram}}(\mathbb{R})$ is the semigroup of measures whose moment generating function is finite in a neighborhood of zero (Cramér's condition).

For those semigroups listed here that strictly contain $\mathscr{L}^1$, we have shown that $\mathscr{L}^\psi_0$ and $\mathscr{L}^p$ for any $p \in (0,1)$ (including the limit case $\mathscr{L}^0 = \mathcal{P}(\mathbb{R})$) only admit the zero monotone homomorphism. However, the intermediate semigroups $\mathscr{L}^1_\zeta$, $\mathscr{L}^\psi_=$, $\mathscr{L}^\psi_\pm$ and $\mathscr{L}^\psi$ have non-trivial monotone homomorphisms, and some have arguably large sets thereof.

For those semigroups listed above that are contained in $\mathscr{L}^1$, including $\mathscr{L}^1$ itself, we have shown that the only monotone homomorphisms are multiples of the expectation. Of course there are also intermediate semigroups that admit other monotone homomorphisms, for example the one where $\lim_{n\to\infty} n \log^2 n \cdot \mu((n,\infty))$ exists and is finite.[2] This semigroup (analogous to $\mathscr{L}^\psi$) lies strictly between $\mathscr{L}^q$ for $q > 1$ and $\mathscr{L}^1$, and has

$$\mu \longmapsto \lim_{n\to\infty} n \log^2 n \cdot \mu((n,\infty))$$

as a monotone homomorphism.

6. Open questions

We end the paper with a number of open questions.

- The variance is a homomorphism which instead of monotonicity has the related property of non-negativity. Is there a (non-trivial) non-negative homomorphism on the entirety of $\mathcal{P}(\mathbb{R})$?
- Mattner [5] shows that the variance is the unique continuous non-negative homomorphism on $\bigcap_p \mathscr{L}^p$. Are there non-continuous ones?
- The stochastic order can be defined for $\mathbb{R}^d$. What are the monotone homomorphisms from $\mathcal{P}(\mathbb{R}^d)$ to $\mathbb{R}$?
- Given $\mu, \nu \in \mathcal{P}(\mathbb{R})$, under what conditions is there $\eta \in \mathcal{P}(\mathbb{R})$ with $\mu * \eta \geq \nu * \eta$? For $\mu, \nu \in \mathscr{L}^1$, it was shown in [8] that if μ and ν are different, then a necessary and sufficient condition is that $\mathbb{E}[\mu] > \mathbb{E}[\nu]$.[3]

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[2] This is contained in $\mathscr{L}^1$ because $\sum_{n=2}^\infty \frac{1}{n \log^2 n} < \infty$.

[3] Sufficiency is Theorem 1 in [8]; the end of their Appendix A shows necessity.

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