A QUANTITATIVE NEUMANN LEMMA FOR FINITELY GENERATED GROUPS

ELIA GOROKHOVSKY, NICOLÁS MATTE BON, AND OMER TAMUZ

ABSTRACT. We study the coset covering function $\mathfrak{C}(r)$ of an infinite, finitely generated group: the number of cosets of infinite index subgroups needed to cover the ball of radius r. We show that $\mathfrak{C}(r)$ is of order at least $\sqrt{r}$ for all groups. Moreover, we show that $\mathfrak{C}(r)$ is linear for a class of amenable groups including virtually nilpotent and polycyclic groups, and that it is exponential for property (T) groups.

1. INTRODUCTION

Let G be an infinite discrete group generated by a finite symmetric set S. We say that a coset $C = Hg$ is a coset of infinite index (c.i.i.) of G if H is an infinite index subgroup of G. Neumann's Lemma [13] states that G cannot be covered by a finite number of c.i.i.s. We refine this question, and ask: how many c.i.i.s are needed to cover $B_r(G, S)$, the ball of radius r in G? For each r, denote by $\mathfrak{C}(r)$ the smallest number of c.i.i.s needed to cover B_r. That is,

$$\mathfrak{C}_{G,S}(r) = \min\left\{N \ : \ \exists \text{ c.i.i.s } C_1, \ldots, C_N \text{ s.t. } B_r(G, S) \subseteq \cup_{i=1}^N C_i\right\}.$$

We call $\mathfrak{C}_{G,S}$ the *coset covering function* of (G, S).

A number of natural questions arise: What lower and upper bounds on the coset covering function apply to all finitely generated groups? And how can these bounds be improved for groups with particular algebraic or geometric properties? The goal of this note is to establish some results in this direction, as well as to advertise the study of this invariant.

Neumann's Lemma suggests (but does not immediately imply) that the coset covering function should tend to infinity with r for any infinite

group G. Our first result confirms this intuition, and moreover provides a quantitative lower bound that applies to all groups.

Theorem 1. *For any infinite, finitely generated group (G, S) with symmetric S, we have $\mathfrak{C}_{G,S}(r) \geq \sqrt{r}/(4|S|)$.*

The proof of this claim is a probabilistic argument that relies on the analysis of a random walk on the group, and follows from a result of Lyons [12]. Using the same argument, the universal lower bound in Theorem 1 can be improved when the distance from the origin of the random walk satisfies some upper bound with positive probability; see Theorem 3.1.

Consider the smallest infinite group $G = \mathbb{Z}$ and the standard generating set. Here $\mathfrak{C}_{G,S}(r) = 2r + 1$, since the only infinite index subgroup is the trivial one. Our next result shows that the coset covering function grows linearly for a class of amenable groups, namely all groups that admit a "cautious random walk" in the terminolgy of Erschler and Ozawa [6]. We recall the definition below and mention here that, by results of Tessera [15, 16] and Erschler–Zheng [7] this class includes virtually nilpotent and polycyclic groups, solvable Baumslag-Solitar groups, lamplighter groups $F \wr \mathbb{Z}$ where F is a finite group, and solvable groups of finite Prüfer rank.

Theorem 2. *For any infinite, finitely generated, group (G, S) that admits a cautious random walk there exist constants $c_1, c_2 > 0$ such that*

$$c_1 r \leq \mathfrak{C}_{G,S}(r) \leq c_2 r.$$

The upper bound in the previous theorem is a straightforward application of a result of Erschler and Ozawa [6], who showed that groups with a cautious random walk have Shalom's property H_{FD}, which for amenable groups implies the existence of a finite index subgroup which surjects onto $\mathbb{Z}$ [14]. It is indeed immediate to see that $\mathfrak{C}_{G,S}(r) \leq cr$ for any group which virtually surjects onto $\mathbb{Z}$ (see Claims 2.1-2.2). Note moreover that in all the special cases mentioned above, the existence of a virtual surjection to $\mathbb{Z}$ is obvious without appealing to Shalom's property. The main point of Theorem 2 is that the same assumption also implies a matching lower bound, as a consequence of Theorem 3.1. For virtually nilpotent groups, we provide an alternative proof in §3.3 which does not rely on random walks.

Finally, we note that the function $\mathfrak{C}_{G,S}(r)$ is always bounded above by an exponential function, since the ball $B_r(G, S)$ can be always covered by points, i.e. cosets of the trivial subgroup. Our last result shows that for some groups, the function $\mathfrak{C}_{G,S}(r)$ may actually grow exponentially.

Theorem 3. *For any infinite, finitely generated group (G, S) with property (T) there exists a constant ε such that*

$$\mathfrak{C}_{G,S}(r) \geq e^{\varepsilon r}.$$

The constant ε can be bounded from below by a monotone transformation of the Kazhdan constant of (G, S).

Open questions. We conclude this introduction by suggesting some problems for further investigation.

We are not aware of any examples in which the coset covering function achieves the bound of Theorem 1, or indeed of any in which it is sub-linear. Thus the following natural question is open:

Question 4. *Does there exist an infinite, finitely generated group (G, S) such that $\liminf_r \frac{1}{r}\mathfrak{C}_{G,S}(r) = 0$?*

Note that by Claim 2.2 below, it is equivalent to consider free groups, because if any group with d generators has a small covering function, then so does the free group with d generators. Moreover, by Claim 2.1, it suffices to consider the free group with two generators, since it contains every finitely generated free group with more generators as a finite index subgroup.

One difficulty in estimating the coset covering function is to control the cosets of different subgroups at the same time. However in some situations the cardinality of a minimal cover of $B_r(G, S)$ by c.i.i.s can be achieved (up to constants) by considering only covers by cosets of a single subgroup infinite index (s.i.i.): this is the case for the groups covered by Theorems 2 and 3. It is natural to wonder whether this is a general fact. To formalise this, let us introduce the following invariant

$$\mathfrak{D}_{G,S}(r) = \min \left\{ N \; : \; \exists H \text{ s.i.i and } g_1, \ldots, g_N \text{ s.t. } B_r(G, S) \subseteq \cup_{i=1}^{N} g_i H \right\}.$$

Note that $\mathfrak{D}_{G,S}(r)$ has a geometric interpretation: it is the cardinality of the smallest ball of radius r in the Schreier graph of G/H, when H varies among all s.i.i.s of G (in contrast, $\mathfrak{C}_{G,S}$ lacks such an interpretation). In particular $\mathfrak{D}_{G,S}(r) \geq r$ trivially. We clearly have

$$\mathfrak{C}_{G,S}(r) \leq \mathfrak{D}_{G,S}(r)$$

so that we are lead to ask if the previous inequality can be strict asymptotically.

Question 5. *Does there exists an infinite, finitely generated group (G, S) such that $\liminf_{r \to \infty} \frac{\mathfrak{C}_{G,S}(r)}{\mathfrak{D}_{G,S}(cr)} = +\infty$ for every $c > 0$?*

Note that any example (G, S) providing an affirmative answer to Question 4 would necessarily give an affirmative answer to Question 5.

We propose the following concrete problem as a test case for the above questions.

Problem 6. *Estimate the coset covering function of the Grigorchuk group.*

Note that for the Grigorchuk group with its standard generating set we have $\mathfrak{D}_{G,S}(r) = r + 1$, as follows from the fact that it admits a Schreier graph isometric to a ray [1].

Finally we observed that in all examples in this paper for which the coset covering function can been computed, it grows either linearly (as in Theorem 2) or exponentially (as in Theorem 3). We thus ask:

Problem 7. *Find a group for which the coset covering function is neither linear nor exponential.*

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2. Preliminaries

To simplify notation, define the equivalence relation $\sim$ on the space of functions $\mathbb{N} \to \mathbb{N}$ by $f \sim g$ if and only if there exist constants $c_1, c_2 > 0$ such that

$$c_1 f(c_1 r) \leq g(r) \leq c_2 f(c_2 r) \quad \text{for all } r.$$

A standard argument shows that if S and T generate G then $\mathfrak{C}_{G,S} \sim \mathfrak{C}_{G,T}$. Accordingly, when we consider $\mathfrak{C}_{G,S}$ up to equivalence, we will write $\mathfrak{C}_G$ and omit the generating set.

2.1. The covering function of finite index subgroups and of quotients.

Claim 2.1. *If $[G : H] < \infty$, then $\mathfrak{C}_G \sim \mathfrak{C}_H$.*

Proof. Let S be a generating set for H and $T \supseteq S$ a generating set for G. We will show that

$$(2.1) \qquad \mathfrak{C}_{H,S}(r) \leq \mathfrak{C}_{G,T}(r) \leq [G : H]\mathfrak{C}_{H,S}(Cr + D),$$

for some $C, D > 0$. Fix r. Let $\ell = [G : H]$ and let $Hg_1, \ldots, Hg_\ell$ be the cosets of H in G.

To prove the first inequality in (2.1) we will construct a covering of $B_r(H, S)$ by $\mathfrak{C}_{G,T}(r)$ c.i.i.s.

Denote by $|\cdot|_S$ and $|\cdot|_T$ word lengths with respect to S in H and T in G, respectively. Note that for any $h \in H$, we have $|h|_T \leq |h|_S$, since $S \subseteq T$. Therefore $B_r(H,S) \subseteq B_r(G,T)$. Suppose $B_r(G,T) \subseteq \bigcup_{i=1}^m K_i f_i$, where each $K_i f_i$ is a c.i.i. in G. Then

$$B_r(H,S) \subseteq B_r(G,T) \cap H \subseteq H \cap \bigcup_{i=1}^m K_i f_i = \bigcup_{i=1}^m (H \cap K_i) h_i$$

for some $h_1, \ldots, h_m \in H$. Moreover, $[H : H \cap K_i] = \infty$, and each $(H \cap K_i) h_i$ is a c.i.i, and $\mathfrak{C}_{H,S}(r) \leq \mathfrak{C}_{G,T}(r)$.

We now turn to the second inequality in (2.1). Since the inclusion of (H,S) inside (G,T) is a quasi-isometry, we can fix a constant $C > 0$ such that $H \cap B_r(G,T) \subseteq B_{Cr+C}(H,S)$. We will construct a covering of $B_r(G,T)$ by $[G : H]\mathfrak{C}_{H,S}(Cr + D)$ c.i.i.s, where $D = C(k + 1)$ and $k = \max_{1 \leq i \leq \ell} |g_i|_T$.

Let $m = \mathfrak{C}_{H,S}(Cr + D)$, and let $B_{Cr+D}(H,S) \subseteq \bigcup_{i=1}^m K_i h_i$, with each $K_i h_i$ a c.i.i. in H. Then

$$B_r(G,T) = G \cap B_r(G,T)$$
$$= \left(\bigcup_{i=1}^\ell H g_i \right) \cap B_r(G,T)$$
$$= \bigcup_{i=1}^\ell (H \cap B_r(G,T) g_i^{-1}) g_i.$$

Because $g_i \in B_k(G,T)$, $B_r(G,T) g_i^{-1} \subseteq B_{r+k}(G,T)$. On the other hand,

$$H \cap B_{r+k}(G,T) \subseteq B_{C(r+k)+C}(H,S) = B_{Cr+D}(H,S) \subseteq \bigcup_{i=1}^m K_i h_i.$$

Therefore

$$B_r(G,T) \subseteq \bigcup_{i=1}^\ell (H \cap B_{r+k}(G,T)) g_i \subseteq \bigcup_{i=1}^\ell \left(\bigcup_{j=1}^m K_j h_j \right) g_i = \bigcup_{i,j} K_j h_j g_i.$$

The number of cosets in this cover is at most $\ell m = [G : H]\mathfrak{C}_{H,S}(Cr + D)$. $\qquad \square$

Claim 2.2. *If $Q = G/N$ is a quotient of G and $\varphi \colon G \to Q$ is the quotient map, then*

$$\mathfrak{C}_{G,S}(r) \leq \mathfrak{C}_{Q,\varphi(S)}(r).$$

Proof. If $B_r(Q, \varphi(S))$ is covered by the c.i.i.s $C_1, \ldots, C_m$, then $B_r(G,S)$ is covered by the c.i.i.s $\varphi^{-1}(C_1), \ldots, \varphi^{-1}(C_m)$.

$\qquad \square$

2.2. **Random walks on groups.** Given a finitely generated G, let μ be a finitely supported, symmetric, non-degenerate probability measure on G; the latter condition means that the support of μ generates G as a semigroup. Denote $\min \mu = \min\{\mu(g) : \mu(g) > 0\}$. Let $X_1, X_2, \ldots$ be i.i.d. random variables with distribution μ. Let $Z_n = X_1 \cdot X_2 \cdots X_n$ be the μ-random walk on G.

We will need the following claim, which is a simple consequence of [12, Lemma 3.4].

Claim 2.3. *Let (G, μ) be as above. Let H be an infinite index subgroup of G. Then for any coset Hg and any n,*

$$\mathbb{P}\left[Z_n \in Hg\right] \leq \frac{4}{\min \mu} \cdot \frac{1}{\sqrt{n}}.$$

Proof. Lyons [12] considers Markov chains on a countable state space X, that are reversible with respect to an infinite, positive measure π. Denote the transition matrix of such a chain by $Q(\cdot, \cdot)$, and let $c = \inf\{\pi(x)Q(x, y) : x \neq y \text{ and } Q(x, y) > 0\}$. He shows that if such a chain starts at $x \in X$, then the probability that it is at any particular $y \in X$ at time n can be bounded from above by $\frac{4\pi(x)}{c\sqrt{n+1}}$ (See Lemma 3.4 and remarks 3 and 4).

The process $(HZ_n)_n$ is well known to be a Markov chain on the coset space $H\backslash G$. This Markov chain is furthermore easily seen to be reversible and to have an infinite uniform stationary measure $\pi(x) = 1$; the infinitude of this measure is due to H having infinite index in G. The transition probabilities are at least $\min \mu$ whenever they are non-zero, and thus this chain satisfies the conditions of the above mentioned result, and we can conclude that the probability that this Markov chain occupies any particular state at time n is at most $\frac{4}{\min \mu} \frac{1}{\sqrt{n+1}}$, which (to simplify the exposition) is at most $\frac{4}{\min \mu} \frac{1}{\sqrt{n}}$. $\qquad \square$

3. Proofs

We start with a proof of Theorem 1. The idea of the proof is to consider a simple random walk on the group, and to show that if the set of c.i.i.s is too small then the random walk misses it with positive probability.

Proof of Theorem 1. Let $H_1g_1, H_2g_2, \ldots, H_Ng_N$ be cosets of infinite index of G.

Let μ be the uniform distribution on S, so that $\min \mu = 1/|S|$. Let $(Z_n)_n$ be the μ-random walk. It follows from Claim 2.3 that for every

i,

$$\mathbb{P}\left[Z_n \in H_i g_i\right] \leq \frac{4|S|}{\sqrt{n}}.$$

By the union bound

$$\mathbb{P}\left[Z_n \in \cup_i H_i g_i\right] \leq \frac{4|S|N}{\sqrt{n}}.$$

It follows that if $N < \frac{\sqrt{n}}{4|S|}$ then with positive probability Z_n is not in $\cup_i H_i g_i$. Since $Z_n \in B_n$ with probability 1, it follows that at least $\frac{\sqrt{r}}{4|S|}$ c.i.i.s are needed to cover the ball of radius r.

$\square$

Our next result uses the same idea to provide sharper lower bounds on $\mathfrak{C}_{G,S}$ for groups in which one can bound from above the distance of a random walk from the origin. Given a monotone increasing function $f \colon \mathbb{N} \to \mathbb{N}$, we write

(3.1) $$f^*(n) = \max\{k \colon f(k) \leq n\}.$$

Theorem 3.1. *Let (Z_n) be a finitely supported, symmetric, non-degenerate μ-random walk on G. Denote $S = \operatorname{supp} \mu$. Assume that $f \colon \mathbb{N} \to \mathbb{N}$ is a function such that*

$$\mathbb{P}[Z_n \in B_{f(n)}(G, S)] > c$$

for some $c > 0$. Then

$$\mathfrak{C}_{G,S}(r) \geq \frac{c \min \mu}{4} \sqrt{f^*(r)}.$$

Proof. Let $H_1 g_1, H_2 g_2, \ldots, H_N g_N$ be cosets of infinite index of G, and suppose that they cover the ball $B_{f(n)}(G, S)$, with $N = \mathfrak{C}_{G,S}(f(n))$. Then the event $\{Z_n \in B_{f(n)}(G, S)\}$ is contained in $\bigcup_{i=1}^{N}\{Z_n \in H_i g_i\}$. On the other hand, it follows from Claim 2.3 that for every i,

$$\mathbb{P}\left[Z_n \in H_i g_i\right] \leq \frac{4}{\min \mu \sqrt{n}}.$$

Thus, by the union bound

$$c \leq \mathbb{P}\left[Z_n \in B_{f(n)}(G, S)\right] \leq \frac{4N}{\min \mu \sqrt{n}}.$$

It follows that $\mathfrak{C}_{G,S}(f(n)) = N > \frac{c \min \mu}{4}\sqrt{n}$. Set $n = f^*(r)$. Then $f(n) \leq r$ and since $\mathfrak{C}_{G,S}$ is monotone increasing we obtain

$$\mathfrak{C}_{G,S}(r) \geq \mathfrak{C}_{G,S}(f(n)) \geq \frac{c \min \mu}{4}\sqrt{f^*(r)},$$

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as desired. □

Recall that the speed of the random walk (Z_n) is the function

$$L_S(n) = \mathbb{E}\left[|Z_n|_S\right],$$

where $|\cdot|_S$ is the word norm on (G, S). Then Theorem 3.1 readily implies the following.

Corollary 3.2. *We have*

$$\mathfrak{C}_{G,S}(r) \geq \frac{L_S^*(\lfloor r/2\rfloor)^{\frac{1}{2}}}{8|S|},$$

where L_S^* is defined as in (3.1).

Proof. Let μ be the uniform distribution on S. By the Markov inequality, we have

$$\mathbb{P}\left[Z_n \in B_{2L_S(n)}(G, S)\right] \geq \frac{1}{2}.$$

Thus we may apply Theorem 3.1 to the function $f(n) = 2L_S(n)$, with $c = 1/2$. Since $f^*(n) \geq L_S^*(\lfloor \frac{n}{2}\rfloor)$, the inequality in the statement follows. □

Remark 3.3. . *The lower bound provided by Theorem 3.1 is interesting (i.e. sharper than the general $\sqrt{n}$ bound in Theorem 1) precisely if the function $f(n)$ in the statement satisfies $f(n)/n \to 0$ along a subsequence. The existence of such a function is equivalent to the triviality of the Poisson boundary of the random walk induced by μ (also called the Liouville property): one direction follows from the entropy criterion of Kaimanovich–Vershik [11] and Derrienic [5], and the other from Varopoulous' estimate [17]. The triviality of the Poisson boundary implies in particular that the group G is amenable.*

Remark 3.4. *Corollary 3.2 is a convenient criterion to apply Theorem 3.1, as the speed of the random walk has been computed in many cases. However we point out that it is a-priori easier to find a function $f(n)$ satisfying the assumption of Theorem 3.1 than to bound from below the speed of the random walk.*

3.1. **Groups that admit a cautious random walk.** Theorem 3.1 provides a criterion, in terms of the random walk, under which the coset covering function must admit a linear lower bound: this holds true as long as the assumption of the theorem is satisfied for a function $f(n) \sim \sqrt{n}$. Perhaps surprisingly, a tightly related condition also implies a matching linear upper bound, thanks to results of Erschler and Ozawa [6] and Shalom [14]. Following [6, 8], we give the following definition.

Definition 3.5. Let G be a finitely generated group and μ a symmetric, non-degenerate probability measure μ on G with finite support S. We say that the random walk (Z_n) on (G, μ) is *cautious* if for every $\varepsilon > 0$ we have

$$\inf_n \mathbb{P}\left[Z_n \in B_{\varepsilon\sqrt{n}}(G, S)\right] > 0.$$

We will further say that a finitely generated group G *admits a cautious random walk* if there exists μ with the above properties such that the random walk on (G, μ) is cautious.

This property first appeared in [6, Corollary 2.5], although the reader should be warned that the terminology "cautious random walk" is used with slightly different meanings in [6–8]; the definition given here coincides with that of [8].

It follows from the results of Hebisch and Saloff-Coste [10] that if G has polynomial growth then it admits a cautious random walk. Erschler and Zheng prove in [7, Lemma 4.5] that the simple random walk is cautious for every group G which admits controlled Følner pairs in the sense of Tessera [15]. We shall not need the definition of controlled Følner pairs, but we mention that by results of Tessera [15, 16] the class of groups admitting controlled Følner pairs includes all polycyclic groups, Baumslag-Solitar groups, lamplighter groups $F \wr \mathbb{Z}$ where F is a finite group, and solvable groups of finite Prüfer rank.

Erschler and Ozawa show that a group that admits a cautious random walk has Shalom's property H_{FD} [6, Corollary 2.5]. Shalom shows that if G is amenable and has property H_{FD} then it virtually surjects to $\mathbb{Z}$ [14, Theorem 4.3.1]. Since groups that admit a cautious random walk are amenable (see Remark 3.3), these results implies the following.

Theorem 3.6 (Erschler and Ozawa, Shalom). *Let G be a finitely generated infinite group which admits a cautious random walk. Then G has a finite index subgroup which surjects onto $\mathbb{Z}$.*

By Claims 2.1-2.2, this implies that if G admits a cautious simple random walk, then $\mathfrak{C}_{G,S}(n) \leq Cn$ for some $C > 0$. Combining with Theorem 3.1, we obtain Theorem 2. In §3.3 below we provide an elementary proof of this theorem, for virtually nilpotent groups.

3.2. **Groups with property (T).** In this section we prove Theorem 3. Without loss of generality, we can assume that the generating set S contains the identity. Let $H_1 g_1, H_2 g_2, \ldots, H_N g_N$ be cosets of infinite index of G. For $i \in \{1, \ldots, N\}$, let $P \colon \ell^2(H_i \backslash G) \to \ell^2(H_i \backslash G)$ be

the Markov operator given by

$$[Pf](H_ig) = \frac{1}{|S|} \sum_{s \in S} f(H_igs).$$

Since G has property (T) and $1 \in S$, there is an $\varepsilon > 0$ such that the operator norm of P is at most $e^{-\varepsilon}$, uniformly over all infinite index subgroups H (see, e.g., [3, Lemma 12.1.9]). Let $f_n \in \ell^2(H_i \backslash G)$ be given by

$$f_n(H_ig) = \mathbb{P}[H_iZ_n = H_ig],$$

and note that $f_{n+1} = Pf_n$. It follows that the ℓ^2-norm of f_n is at most $e^{-\varepsilon n}$, since $f_0 = \delta_e$ has norm 1. Thus

$$\mathbb{P}[Z_n \in H_ig_i] = f_n(H_ig_i) \le e^{-\varepsilon n}.$$

By the union bound

$$\mathbb{P}[Z_n \in \cup_i H_ig_i] \le Ne^{-\varepsilon n}.$$

It follows that if $N < e^{\varepsilon n}$ then with positive probability Z_n is not in $\cup_i H_ig_i$, and since $Z_n \in B_n$ with probability 1, at least $e^{\varepsilon r}$ c.i.i.s are needed to cover the ball of radius r.

Remark 3.7. *The assumption that G has property (T) can be weakened, as the above proof of Theorem 3 only requires the existence of a unifom spectral gap for the quasi-regular representations $G \curvearrowright \ell^2(H \backslash G)$ for $H < G$ of infinite index. This property is equivalent to property FM in the sense of Cornulier [4], namely that every action of G on a set which preserves an invariant mean has a finite orbit. Property FM is implied by property (T), but it is weaker in general.*

3.3. **Virtually nilpotent groups.** In this section we provide an elementary proof of Theorem 2, for the case of virtually nilpotent groups. The next proposition, which is the core of the proof of this theorem, shows that the so-called "doubling condition" implies a linear lower bound on the coset covering function. The idea of the proof is to get an upper bound on the size of the intersection of a c.i.i. with $B_r(G, S)$.

Proposition 3.8. *Suppose that there exists some $L > 0$ and r_0 such that for all $r > r_0$ we have*

$$\frac{|B_{2r}(G, S)|}{|B_r(G, S)|} \le L.$$

Then for all $r > r_0$,

$$\mathfrak{C}_{G,S}(r) \ge \frac{r}{L}.$$

Proof. Write the generating set as $S = \{s_1, \ldots, s_m\}$, and fix $r > r_0$.

First, notice that $|B_r(G, S) \cap C| \leq |B_{r+1}(G, S) \cap Cs_i|$ for any r, any c.i.i. C (in fact, for any subset of G) and any generator s_i. This holds because $(B_r(G, S) \cap C)s_i$ is a subset of $B_{r+1}(G, S) \cap Cs_i$.

Consider the Schreier coset graph of an infinite-index subgroup H in G. This is a graph whose nodes are (right) cosets of H, which has an edge between two cosets if one can be obtained from the other by right multiplication by an element of S. Since the Schreier graph of H in G is infinite (H has infinite index), it contains arbitrarily long paths. Find a path of length r starting at some $C = Hg$, with its edges marked by generators $s_{i_1}, s_{i_2}, \ldots, s_{i_r}$. Denote $g_k = s_{i_1} s_{i_2} \cdots s_{i_k}$. The nodes on this path are $C, Cg_1, Cg_2, \ldots, Cg_r$.

From the argument above, we have $|B_{r+1}(G, S) \cap Cg_1| \geq |B_r(G, S) \cap C|$. Similarly,

$$|B_{r+k}(G, S) \cap Cg_k| \geq |B_{r+k-1}(G, S) \cap Cg_{k-1}| \geq |B_r(G, S) \cap C|$$

for any $k \leq r$.

Moreover $|B_{2r}(G, S) \cap Cg_k| \geq |B_{r+k}(G, S) \cap Cg_k|$ for $k \leq r$, and so

$$|B_{2r}(G, S) \cap Cg_k| \geq |B_r(G, S) \cap C|.$$

Taking the sum from $k = 1$ to $k = r$ yields

$$\sum_{k=1}^{r} |B_{2r}(G, S) \cap Cg_k| \geq r|B_r(G, S) \cap C|.$$

Since each Cg_k is a distinct coset of H, they are all disjoint. So,

$$\sum_{k=1}^{r} |B_{2r}(G, S) \cap Cg_k| = \left| B_{2r}(G, S) \cap \bigcup_{i=1}^{r} Cg_k \right| \leq |B_{2r}(G, S)|.$$

Therefore,

$$|B_r(G, S) \cap C| \leq \frac{1}{r}|B_{2r}(G, S)| = \frac{1}{r}\frac{|B_{2r}(G, S)|}{|B_r(G, S)|}|B_r(G, S)|.$$

Applying the doubling property now yields

$$|B_r(G, S) \cap C| \leq \frac{L}{r}|B_r(G, S)|.$$

We have thus shown that each c.i.i. takes at most a $\frac{L}{r}$-fraction of the ball of radius r. Hence at least $\frac{r}{L}$ c.i.i.s are needed to cover the ball. $\qed$

Given this proposition, the proof of Theorem 2 for virtually nilpotent groups is straightforward.

It is a well-known consequence of the result of Bass [2] and Guivarc'h [9] that every virtually nilpotent group satisfies the doubling condition in the hypothesis of Proposition 3.8, and thus for each such group (G, S) there is a constant $c > 0$ such that $\mathfrak{C}_{G,S}(r) \geq cr$. For the other direction, note that every infinite nilpotent group has a quotient to $\mathbb{Z}$, and thus, by Claim 2.2 there is another constant c' such that $\mathfrak{C}_{G,S}(r) \leq c'r$. Claim 2.1 now implies that the same holds for every infinite virtually nilpotent group. This completes the proof.

References

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California Institute of Technology

CNRS, Institut Camille Jordan (ICJ, UMR CNRS 5208), Université Lyon 1