

TAXES AND MARKET POWER: A NETWORK APPROACH

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ABSTRACT. Suppliers of differentiated goods make simultaneous pricing decisions, which are strategically linked due to consumer preferences and the structure of production. Because of market power, the equilibrium is inefficient. We study how a policymaker should target a budget-balanced tax-and-subsidy policy to increase welfare. A key tool is a certain basis for the goods space, determined by the network of interactions among suppliers. It consists of *eigenbundles*—orthogonal in the sense that a tax on any eigenbundle passes through *only* to its own price—with pass-through coefficients determined by associated eigenvalues. Our basis permits a simple characterization of optimal interventions. For example, a planner maximizing consumer welfare should tax eigenbundles with low pass-through and subsidize ones with high pass-through. We interpret these results in terms of the network structure of the market.

1. INTRODUCTION

Market power and strategic pricing are significant in many intermediate and final good markets. On the supply side, the technology of production creates strategic linkages: for example, when two intermediates are present in the same final goods, an increase in one's price reduces the other's demand.¹ On the demand side, the substitution patterns between goods also directly shape which firms are in competition. The structure of such relationships determines how producers price, how surplus is allocated among them, and the welfare implications of markets for consumers.

This paper presents a simple model in which different combinations of intermediate inputs go into producing a set of final goods, and the demands of these final goods arise from the behavior of a representative consumer. Suppliers choose prices simultaneously. A special case of this model is a differentiated oligopoly. Another special case is a supply network in which suppliers are interrelated because they are producing inputs for the same final goods, which have independent demands. We study the Nash equilibrium of this pricing game. As suppliers have market power, the equilibrium is typically inefficient.² This creates the scope for targeted interventions that tax some suppliers and subsidize others to further a social objective. We develop the analysis

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¹For recent contributions to the theory of production and intermediation networks, see [Acemoglu, Carvalho, Ozdaglar, and Tahbaz-Salehi \(2012\)](#), [Antràs and Chor \(2013\)](#), [Bimpikis et al. \(2014\)](#), [Choi, Galeotti, and Goyal \(2017\)](#), [Manea \(2018\)](#), and [Baqae \(2018\)](#).

²For empirical evidence on the size of the welfare loss see [Pelligrino \(2021\)](#) and [Ederer and Pelligrino \(2021\)](#).

focusing on the objective of consumer surplus, but our results can be extended to other objectives such as producer or aggregate surplus.

In the markets we study, the implications of taxing or subsidizing the suppliers are complicated, since changes to firms' costs affect the prices and quantities of other firms' through the network of relationships. We provide a compact and tractable representation of these spillovers, permitting simple formulae for how cost changes pass through to equilibrium prices. In particular, the interactions among suppliers are summarized in a *spillover matrix*, each entry of which reflects the strategic interaction between two firms. This matrix induces a basis of eigenvectors (also called principal components) of the goods space;³ we call the vectors in this basis *eigenbundles*.

The basis has three special properties. First, when the cost of one eigenbundle is changed, the effect is to change equilibrium prices *only of that eigenbundle*. These eigenbundles therefore identify independent, or orthogonal, dimensions of the market, such that the costs of one eigenbundle do not affect the prices of others. Second, the pass-throughs associated with various eigenbundles can be calculated in terms of corresponding eigenvalues of the spillover matrix. Third, the market induces a ranking of pass-throughs: the eigenbundles with larger eigenvalues—which are more representative (in a precise sense) of the input requirements of the market—are those with smaller pass-throughs.

These properties permit us to express the effect of tax-subsidy schemes on prices and on welfare in a form that facilitates a simple characterization of interventions that maximize consumer surplus. Through the lens of our decomposition, the optimal policy may be described as follows: it collects tax revenue from the eigenbundles with low pass-through, where the impact on prices and output is relatively small, and allocates them—via subsidies—to the eigenbundles with high pass-through, where the impact on prices and output is relatively large.

1.1. Related literature. Our paper contributes to a literature on the structure and theoretical properties of market power. For an early theoretical paper see [Dixit \(1986\)](#); more recent studies include, for example, [Vives \(1999\)](#) and [Azar and Vives \(2021\)](#). A recent literature in macroeconomics and industrial organization uses network models of differentiated oligopoly, with models similar to the one studied here, to provide empirical estimates of welfare losses due to market power (see e.g., [Pelligrino \(2021\)](#) and [Ederer and Pelligrino \(2021\)](#)).

Our paper makes two contributions to this literature. First, taking a network perspective, we provide a geometric approach to analyzing pass-through in the pricing game with market power. This builds on work emphasizing the value of pass-through as a conceptual tool (e.g., [Weyl and Fabinger \(2013\)](#) and [Miklos-Thal and Shaffer \(2021\)](#)), and shows how it can be described tractably and intuitively in markets with very rich heterogeneity. Second, motivated by the empirical research on welfare losses due to market power, we apply our decomposition to characterize policy interventions that maximize consumer surplus and shed light on the economic forces that make them most effective.

The strategic interactions in the model may come from either the consumer's preferences or the structure of technology. The latter interpretation connects our work to literature on input-output models. The canonical models in this literature (e.g., [Acemoglu, Carvalho, Ozdaglar, and Tahbaz-Salehi \(2012\)](#)) assume competitive markets.

³That is, the space of goods produced by the strategic firms.

A more recent work in this literature examines the role of market power (e.g., Baqaee (2018), Grassi (2017) and Liu (2019) and Grassi and Sauvagnat (2019)). This work highlights the importance of the interaction between market structure and production networks in determining shock amplification and the desirability of appropriate interventions. Our paper highlights the usefulness of a principal component decomposition of cost changes in the construction of optimal balanced-budget interventions to maximize consumer surplus.⁴

More generally, our paper contributes to the theory of network interventions; prominent early contribution to this theory include Borgatti (2006) and Ballester, Calvó-Armengol, and Zenou (2006).⁵ In a recent paper, Galeotti, Golub, and Goyal (2020) study intervention in quadratic games. They use the singular value decomposition of the interaction matrix for the study of (costly) interventions that alter the stand-alone marginal benefits of individual activity. They assume that the costs are separable across individuals and increasing and convex in the magnitude of the intervention (and are independent of the network). By contrast, in this paper, the costs of a subsidy or the revenue of a tax depends on the entire network structure. This difference is crucial; indeed, tax-subsidy intervention and budget-balanced interventions lie outside the scope of Galeotti, Golub, and Goyal (2020). Our results generate new insights: for example, there is a threshold such that a planner subsidizes eigenbundles with eigenvalues below a certain threshold and taxes the other eigenbundles.

2. MODEL

There is a finite set of final goods. Each final good is produced by combining subsets of inputs—possibly in different proportions. Each input is produced by a supplier. The production technology is summarized by a *production network*—a bipartite weighted network whose nodes are suppliers and final goods. This network specifies how much of each input is required to produce one unit of each final good. The combination of demand, prices and the production network determine the demand for inputs.

We start by laying out the basic notation. This is followed by a discussion of how changes in cost pass through to equilibrium prices. We illustrate the determinants of this pass-through in two leading special cases: the case of independent demands (that yields a supply chain example) and the case with a one-to-one mapping from input supplier to final good (that yields the classical differentiated oligopoly model).

Symbols denoting vectors and matrices are in bold. For any matrix M , the symbol m_{ij} stands for its element in the i th row and j th column, and M^T denotes its transpose. The symbol $\langle \mathbf{a}, \mathbf{b} \rangle$ denotes the dot product of \mathbf{a} and \mathbf{b} .

⁴Choi, Galeotti, and Goyal (2017) consider a related model with pure homogeneous products. Condorelli, Galeotti, and Renou (2017) and Manea (2018) study bargaining in networks with intermediation. See also Elliott and Galeotti (2019) for related arguments on how network methods can be useful for competition authorities in developing antitrust investigations.

⁵The literature on this subject is very large. Other contributions of network intervention in models of information diffusion, advertising, and pricing include Banerjee, Chandrasekhar, Duflo, and Jackson (2013), Belhaj and Deroian (2017), Bloch and Querou (2013), Candogan, Bimpikis, and Ozdaglar (2012), Demange (2017), Fainmesser and Galeotti (2017), Galeotti and Goyal (2009), Galeotti and Rogers (2013), and Leduc, Jackson, and Johari (2017).

2.1. Market structure. The set of *final goods* is $\mathcal{F} = \{1, 2, \dots, F\}$; final goods are indexed by f . The set of *inputs* is $\mathcal{N} = \{1, 2, \dots, N\}$. Input i is produced by supplier i . The production network is denoted by \mathbf{T} , an N -by- F matrix with typical element $t_{if} \geq 0$. The interpretation is that the production of one unit of good $f \in \mathcal{F}$ requires t_{if} units of each input i .

After all the suppliers simultaneously choose their prices $\{p_i\}_{i \in \mathcal{N}}$, final goods are produced and priced competitively.⁶ Thus the price of final good f equals its marginal cost of production, which is the sum of the prices of the inputs needed to produce one unit of the final good. That is, final goods' prices are

$$P(\mathbf{p}) = \mathbf{T}^\top \mathbf{p}.$$

Final demands come from a representative consumer with a utility function $U(\cdot)$ such that, for any given price vector, $\mathbf{P} = (P_f)_{f \in \mathcal{F}}$, the demand profile solves

$$\max_{\mathbf{Q}} U(\mathbf{Q}) - \langle \mathbf{Q}, \mathbf{P} \rangle. \quad (1)$$

We will begin our analysis with a linear-quadratic specification of utility that will give rise to linear demands. However, we will keep some formulas general when possible with a view toward extending the analysis.

The consumer has (gross) utility for consuming a bundle \mathbf{Q} of final goods,

$$U(\mathbf{Q}) = \sum_f \beta_f Q_f - \frac{1}{2} \mathbf{Q}^\top \mathbf{B} \mathbf{Q}, \quad (2)$$

where $\beta_f > 0$ are positive constants and \mathbf{B} is a given positive-definite matrix (Amir, Erickson, and Jin (2017), Choné and Linnemer (2020), and Vives (1999)). Assuming the consumer has sufficient income (a condition we will take for granted), this induces linear demands $Q : \mathbb{R}^F \rightarrow \mathbb{R}^F$ where $Q_f(\mathbf{P})$ is the quantity demanded of good f at the profile of prices \mathbf{P} , with

$$\partial Q_f / \partial P_{f'} = (\mathbf{B}^{-1})_{ff'} \quad \text{and} \quad \frac{\partial^2 Q_f}{\partial P_f \partial P_{f'}} = 0 \quad \text{for any } f, f' \in \mathcal{F}. \quad (3)$$

The associated demand function for inputs is called $q : \mathbb{R}^N \rightarrow \mathbb{R}^N$ and can be written as

$$q(\mathbf{p}) := \mathbf{T} Q(\mathbf{T}^\top \mathbf{p}). \quad (4)$$

When the consumer chooses optimal quantities $\mathbf{Q}(\mathbf{P})$, a standard calculation (plugging in the first-order conditions for optimization) her payoff is

$$U^* = \frac{1}{2} \mathbf{Q}(\mathbf{P})^\top \mathbf{B} \mathbf{Q}(\mathbf{P}). \quad (5)$$

⁶The assumption that final goods are priced competitively allows us to focus on strategic competition between the suppliers. The analysis can be extended to the case in which firms producing final goods also have market power.

2.2. Equilibrium. We focus on (an interior pure-strategy Nash) equilibrium \mathbf{p}^* of the price setting game between the suppliers. Supplier i has an input with a constant marginal cost of c_i . The first order conditions that characterize the equilibrium are:

$$q_i(\mathbf{p}^*) + \frac{\partial q_i(\mathbf{p}^*)}{\partial p_i}(p_i^* - c_i) = 0 \quad \text{for all } i \in \mathcal{N}. \quad (6)$$

We assume that this equilibrium exists and is unique. Slightly abusing terminology, we denote the equilibrium prices \mathbf{p}^* , quantities $q(\mathbf{p}^*)$, final good prices $\mathbf{T}^\top \mathbf{p}^*$, and final good quantities $Q(\mathbf{T}^\top \mathbf{p}^*)$ by \mathbf{p} , \mathbf{q} , \mathbf{P} and \mathbf{Q} , respectively.

We are interested in how an arbitrary change in production costs,

$$\dot{\mathbf{c}} := (\dot{c}_1, \dot{c}_2, \dots, \dot{c}_N),$$

passes through to equilibrium prices, and how it changes welfare outcomes.

Totally differentiating (6) around the equilibrium \mathbf{p} yields:

$$\sum_{j \in \mathcal{N}} \frac{\partial q_i(\mathbf{p})}{\partial p_j} \dot{p}_j + \frac{\partial q_i(\mathbf{p})}{\partial p_i} (\dot{p}_i - \dot{c}_i) = 0. \quad (7)$$

This equation suggests that it will be useful to define the Jacobian of the vector \mathbf{q} in the prices \mathbf{p} ,

$$D_{ij} = \frac{\partial q_i(\mathbf{p})}{\partial p_j}.$$

Then we have

Lemma 1. The Jacobian can be expressed as follows:

$$\mathbf{D} = \mathbf{T}\mathbf{B}^{-1}\mathbf{T}^\top.$$

This follows immediately from Equation 3 characterizing final demand, Equation 4 linking that to intermediate demand, and the chain rule.

We will now introduce a normalization that will be very useful in the rest of our analysis. Note that if we rescale the units of intermediates, letting new units be defined by $\tilde{q}_i = c_i q_i$, then we get a corresponding input requirements matrix $\tilde{\mathbf{T}} = \mathbf{C}\mathbf{T}$, where \mathbf{C} is the matrix with c on the diagonal. This, in turn, is associated with a new Jacobian, $\tilde{\mathbf{D}} = \mathbf{C}\mathbf{T}\mathbf{T}^\top\mathbf{C}$. By choosing \mathbf{C} appropriately, we may therefore make all diagonal entries of $\tilde{\mathbf{D}}$ equal to -1 .

Thus, we may assume the following normalization property without loss of generality:

Property A. $D_{ii} = -1$ for each $i \in \mathcal{N}$

Using this assumption and rearranging Equation 6, we get

$$\dot{p}_i = \dot{c}_i - \sum_{j \in \mathcal{N}} D_{ij} \dot{p}_j \quad (8)$$

By linearity of demand, this formula holds for all changes $\dot{\mathbf{c}}$ when solutions remain interior, not just small changes.

The strategic relations between any two suppliers i and j are captured by the sign of D_{ij} . We shall say that inputs i and j are *strategic substitutes* if D_{ij} is positive and

strategic complements if D_{ij} is negative. The pass-through can be expressed in matrix form as follows:

$$[\mathbf{I} + \mathbf{D}]\dot{\mathbf{p}} = \dot{\mathbf{c}}. \quad (9)$$

2.3. Two examples. We illustrate the scope of the model by discussing two examples.

2.3.1. Differentiated oligopoly. When the production network \mathbf{T} is the identity \mathbf{I} , the pricing game between suppliers boils down to a classical differentiated oligopoly game (e.g., Vives (1999) and Choné and Linnemer (2020)). In particular, we can identify each supplier i with a final good f —whose only input is i —and the demand for supplier i corresponds to the demand for its associated final good f .

The Jacobian matrix $\mathbf{\Delta}$ is defined by $\Delta_{ff'} = \partial Q_f / \partial P_{f'}$. We may assume without loss of generality, by rescaling units, that $\Delta_{ff} = -1$ for each f .⁷ The matrix of strategic interactions among suppliers is then $\mathbf{D} = -\mathbf{\Delta}$ which satisfies Property A.

2.3.2. Supply network. A different case of the model puts the focus on the structure of production, given by \mathbf{T} . To this end, we assume that final good demands are linear, symmetric, and independent, so that $Q_f = 1 - P_f$ for each final good f .

In this case, the Jacobian matrix $\mathbf{\Delta}$, defined by $\Delta_{ff'} = \partial Q_f / \partial P_{f'}$, is equal to $-\mathbf{I}$, where \mathbf{I} denotes the identity matrix. Recall that the demand for inputs is given by $q(\mathbf{p}) = \mathbf{T}Q(\mathbf{T}^\top \mathbf{p})$. So for any input i ,

$$\frac{\partial q_i(\mathbf{p})}{\partial p_j} = \sum_f t_{if} \frac{\partial Q_f(\mathbf{P})}{\partial P_f} \frac{\partial P_f}{\partial p_j} = - \sum_f t_{if} t_{jf},$$

where we have used the assumption that Q_f depends only on P_f (with a slope of -1) to derive the first equality, and the assumption of competitive final good pricing to obtain the second equality. By choosing units appropriately,⁸ we may assume $\sum_f t_{if}^2 = 1$ for all i in \mathcal{N} ; this yields

$$\mathbf{D} = \mathbf{T}\mathbf{T}^\top.$$

Note that \mathbf{D} is a positive and symmetric matrix, and also positive semidefinite, so Property A is satisfied.

Note that, via the production network, all inputs are *complements*—in the sense that, everything else equal, when the price of one of them increases, the quantity demanded of all the others decreases. The strength of this complementarity is $\langle \mathbf{t}_i, \mathbf{t}_j \rangle \in [0, 1]$ and it equals 1 when input i and input j are used exactly in the same way for the production of every final good, whereas it equals 0 when input i and input j are never used together to produce any of the final goods. More generally, the dot product $\langle \mathbf{t}_i, \mathbf{t}_j \rangle$ is a measure of the angle between \mathbf{t}_i and \mathbf{t}_j , called their *cosine similarity*.

It is useful to rewrite (9) as $\dot{\mathbf{p}} = \dot{\mathbf{c}} - \mathbf{T}\mathbf{T}^\top \dot{\mathbf{p}}$. Thus, the complementarity of all inputs in production implies *strategic substitutes* in the pricing game: when the price of one of them increases, the optimal price of any other input j decreases. Here $D_{ij} = \langle \mathbf{t}_i, \mathbf{t}_j \rangle \in [0, 1]$ again gives the strength of this substitutability.

⁷To this end, define $\tilde{Q}_f = c_f Q_f$ and \tilde{P}_f to be the price per new unit. Then $\partial \tilde{Q}_f / \partial \tilde{P}_f = c_f^2 \partial Q_f / \partial P_f$ so by choosing c_f appropriately, one can achieve any desired scaling.

⁸This simply amounts to scaling the units of intermediate i by an appropriate c_i , which scales the sum $\sum_f t_{if}^2 = 1$ by c_i^{-2} .

3. PASS-THROUGH IN TERMS OF EIGENBUNDLES

In this section, we express the pass-throughs of cost changes in a compact way by changing to a convenient basis of *eigenbundles*.

Since it is symmetric, the matrix D is orthogonally diagonalizable. That is, there exists an $N \times N$ orthonormal matrix U such that

$$D = U\Sigma U^\top,$$

where Σ is an $N \times N$ diagonal matrix whose ℓ^{th} diagonal element is the ℓ^{th} -largest eigenvalue of D , called σ_ℓ ; it is nonnegative because D is positive semidefinite. The ℓ^{th} column u^ℓ is the eigenvector of D corresponding to σ_ℓ . We call this the ℓ^{th} *eigenbundle* of D . These vectors have norm 1 and are orthogonal to each other.

The usefulness of the eigenbundles is that—as [Proposition 1](#) below states—a change in the cost of one of them affects only its own prices with a certain coefficient, called the *pass-through*. The pass-throughs are ordered according to their corresponding eigenvalues: the larger is the ℓ^{th} eigenvalue σ_ℓ , the lower is the pass-through from changes in the cost of the ℓ^{th} eigenbundle to its equilibrium price.

For any $x \in \mathbb{R}^N$, let \underline{x} denote $U^\top x$; that is, \underline{x} is the profile x expressed in the basis U . For each $\ell = 1, 2, \dots, N$, we choose the sign of u^ℓ so that $q_\ell \geq 0$.

Proposition 1. Consider any change in costs, $(\dot{c}_1, \dot{c}_2, \dots, \dot{c}_N)$. The change in the equilibrium price of the ℓ^{th} eigenbundle has the following form:

$$\dot{p}_\ell = \lambda_\ell \dot{c}_\ell. \quad (10)$$

where $\lambda_\ell = \frac{1}{1+\sigma_\ell}$ is increasing in ℓ .

Proof. From (10) we get $(I + U\Sigma U^\top) \dot{p} = \dot{c}$. Multiplying both sides by U^\top we get $U^\top (I + U\Sigma U^\top) U U^\top \dot{p} = U^\top \dot{c}$, that is, $\underline{\dot{p}} = (I + \Sigma)^{-1} \underline{\dot{c}}$. \square

The following result calculates cost changes pass through to prices as a function of each supplier's representation in all the eigenbundles.

Corollary 1. The pass-through of a unit increase $\dot{c} = 1_j$ in supplier j 's cost on i 's equilibrium price is

$$\dot{p}_i = \sum_\ell u_j^\ell \lambda_\ell u_i^\ell. \quad (11)$$

Each element of this sum is a product of three terms. First, u_j^ℓ is the effect of supplier j 's cost on the cost of the ℓ^{th} eigenbundle. Second, the cost of this eigenbundle increases its price by a factor λ_ℓ . Third, the increase in the price of ℓ^{th} eigenbundle increases supplier i 's price by u_i^ℓ .

Proof. Since $\dot{p}_i = \sum_\ell u_i^\ell \dot{p}_\ell$, while $\dot{p}_\ell = \lambda_\ell \dot{c}_\ell$, and $\dot{c}_\ell = \sum_j u_j^\ell \dot{c}_j$, we have

$$\dot{p}_i = \sum_\ell u_i^\ell \lambda_\ell \sum_j u_j^\ell \dot{c}_j,$$

which is the desired expression. \square

3.1. The supply network case and the singular value decomposition. In the supply network special case discussed in [subsubsection 2.3.2](#), our decomposition into eigenbundles has an interpretation in terms of the *singular value decomposition* of T , which will prove useful technically.

The *singular value decomposition* (SVD) of T is $T = USV^T$, where

- (a) U is a matrix of orthonormal eigenvectors of TT^T ;
- (b) V is an $F \times F$ matrix whose columns are orthonormal eigenvectors of $T^T T$;
- (c) S is an $N \times F$ diagonal matrix whose ℓ^{th} diagonal entry is $s_\ell = \sqrt{\sigma_\ell}$.

The columns of U , the eigenbundles, are also called the left singular vectors of T .⁹ The columns of V are called the right singular vectors of T . The numbers s_ℓ are called the singular values of T . An interpretation of the SVD is that for each eigenbundle u_ℓ of intermediate goods, it identifies a corresponding bundle v^ℓ of final goods such that $Tv^\ell = s_\ell u_\ell$, allowing us to “translate eigenbundles into the final goods space” if we wish. This will prove useful in some of the proofs.

4. TAXES AND SUBSIDIES

We now examine the ways in which taxes and subsidies can alleviate the inefficiency created by market power. Specifically, we will study the taxes and subsidies that maximize consumers surplus.¹⁰ We develop the basic Pigouvian theory of such interventions through the lens of our network formalism.

Let $\tau = \{\tau_1, \dots, \tau_n\}$ be the profile of per-unit taxes introduced by the planner. In general, the planner may find it optimal to impose positive taxes on some suppliers and negative taxes (that is, subsidies) on others. For concreteness, we assume that the planner must run a balanced budget (i.e., $\langle \tau, q \rangle = 0$).¹¹

The characterizations of pass-through provide a simple way to study the optimal policy. The decomposition of the matrix of strategic interactions provides an ordering of pass-throughs. Equipped with this ordering, a natural guess would be that, in order to maximize consumer surplus, the planner would want to subsidize the high pass-through eigenbundles and tax the low pass-through eigenbundles. As [Proposition 2](#) highlights, this is indeed what the optimal tax scheme does.

Clearly, if the eigenvalue σ_ℓ is the same for all dimensions ℓ with $q_\ell \neq 0$, the planner cannot do anything useful, because all relevant pass-throughs are the same. Let us assume that this is not the case, and let $K = \frac{1}{4} \sum_\ell q_\ell^2 \frac{1+\sigma_\ell}{\sigma_\ell}$. We denote by z the Lagrange multiplier of the budget constraint; this is the shadow price of public funds in our optimization problem.

Proposition 2. The optimal policy taxes each eigenbundle ℓ with $\lambda_\ell < \frac{z}{2K}$ and subsidizes each eigenbundle ℓ with $\lambda_\ell > \frac{z}{2K}$.

⁹When $D = TT^T$, using the singular value decomposition of T yields the form $D = USU^T$, recovering the expression at the beginning of this section.

¹⁰A similar approach can be used to study other instruments (such as taxes on final goods) and other objectives (like producer or total surplus).

¹¹The optimal policy is qualitatively similar if we assume that the budget is not zero.

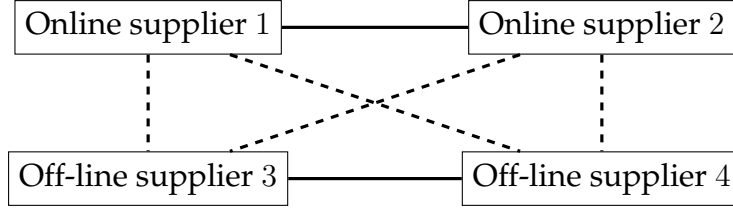


FIGURE 1. Illustration of strategic price relationships within and across on-line and off-line suppliers: Dashed (thick) links denote strategic complements (substitutes).

Proof. As we show in the appendix, the tax on eigenbundle u^ℓ is

$$\frac{1 + \sigma_\ell}{\sqrt{\sigma_\ell}} \cdot \frac{q_\ell}{2} \cdot \frac{z - 2K\lambda_\ell}{z - K\lambda_\ell}, \quad (12)$$

where the shadow price $z \geq 0$ is implicitly defined by

$$\sum_\ell q_\ell^2 \frac{K\lambda_\ell}{\sigma_\ell(z - K\lambda_\ell)^2} = 4.$$

Moreover, for all ℓ with $q_\ell \neq 0$, we have that $z > K\lambda_\ell$, so the sign of (12) coincides with the sign of $z - 2K\lambda_\ell$. \square

5. ILLUSTRATIONS

In this section, we apply our results to describe pass-through and optimal tax-subsidy interventions in the context of two applications—differentiated oligopoly and supply networks.

5.1. Differentiated oligopoly: We present a stylized model of a four-firm differentiated oligopoly. Suppose that each supplier produces one final good. Two of these goods are sold online while the other two are sold off-line. Suppose that a good sold online complements other goods sold online, and is a substitute of goods sold offline. This means that the prices of the two online goods (or the two offline goods) are strategic substitutes, while the prices of an online good and an offline good are strategic complements. Figure 1 represents the network of suppliers with four goods.

We now introduce some numbers for illustration. Let the matrix of strategic interactions among suppliers be given by

$$D = \begin{bmatrix} 1 & 1/2 & -1/2 & -1/2 \\ 1/2 & 1 & -1/2 & -1/2 \\ -1/2 & -1/2 & 1 & 1/2 \\ -1/2 & -1/2 & 1/2 & 1 \end{bmatrix}.$$

The spectral decomposition of this matrix is $U\Sigma U^T$, where (to one decimal place)

$$\Sigma = \begin{bmatrix} 5/2 & 0 & 0 & 0 \\ 0 & 1/2 & 0 & 0 \\ 0 & 0 & 1/2 & 0 \\ 0 & 0 & 0 & 1/2 \end{bmatrix} \quad \text{and} \quad U = \begin{bmatrix} -.5 & -.3 & -.2 & .8 \\ -.5 & -.3 & .8 & -.2 \\ .5 & .3 & .6 & .6 \\ .5 & .9 & 0 & 0 \end{bmatrix}.$$

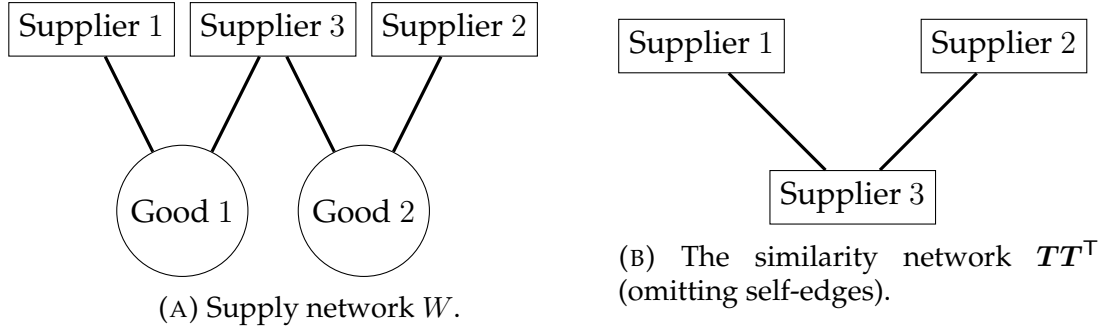


FIGURE 2. From production technology to the similarity network

In this case, Proposition 1 tells us that the pass-through on the first eigenbundle \mathbf{u}^1 is

$$\lambda_1 = \frac{1}{1 + \sigma_1} = \frac{1}{1 + 5/2} = \frac{2}{7} \approx 0.28,$$

while the pass-through on the other eigenbundles \mathbf{u}^ℓ with $\ell > 1$ is

$$\lambda_\ell = \frac{1}{1 + \sigma_\ell} = \frac{1}{1 + 1/2} = \frac{2}{3} \approx 0.67.$$

Hence, by Proposition 2, in order to maximize consumer welfare, the optimal policy taxes the first eigenbundle—the one with relatively low pass through—in order to subsidize the others. To make this concrete, consider the case in which $q_1 = 2$ and $q_2 = q_3 = q_4 = 1$. The optimal policy is as follows, which we give in terms of the eigenbundles but also translate into the original coordinates:

$$\begin{aligned} \underline{q}_1 \tau_1 &= .6 & \text{and} & & q_1 \tau_1 &= .8 \\ \underline{q}_2 \tau_2 &= -.2 & \text{and} & & q_2 \tau_2 &= .5 \\ \underline{q}_3 \tau_3 &= -.1 & \text{and} & & q_3 \tau_3 &= -.7 \\ \underline{q}_4 \tau_4 &= -.3 & \text{and} & & q_4 \tau_4 &= -.7 \end{aligned}$$

The optimal policy taxes the two online suppliers 1 and 2, and subsidizes the offline suppliers 3 and 4. The fact that the optimal policy taxes the largest supplier 1 might seem counterintuitive: Wouldn't it be optimal to try to lower the price of the largest supplier by subsidizing it? The catch is that a per-unit subsidy on the large supplier is relatively expensive. Instead, the optimal policy induces a reduction in the price of supplier 1 indirectly: Taxing supplier 2 leads to a price reduction by the large supplier 1 (due to strategic substitutability between their prices). Similarly, the subsidies on supplier 3 and 4 lead to a lowering of the price of supplier 1 (due to strategic complementarity between 1 and each of 3 and 4). Hence, the optimal policy indirectly induces price reductions by the large supplier 1 via the strategic spillovers caused by the tax on supplier 2 and the subsidies on suppliers 3 and 4. Indeed, the optimal policy achieves this while also raising funds by taxing the large supplier 1.

Using this intuition, we can see that the optimal policy does the opposite when there are instead strategic complementarities within sectors and strategic substitutabilities across sectors. In this case, to induce a reduction in supplier 1's price via indirect effects, the optimal policy taxes the offline sector and subsidizes the small online supplier.

5.2. Supply network. Next consider a supply network with three intermediate good suppliers and two final goods:

$$\mathbf{T} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix} \quad \text{and} \quad \mathbf{T}\mathbf{T}^\top = \begin{bmatrix} 1 & 0 & 1/\sqrt{2} \\ 0 & 1 & 1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} & 1 \end{bmatrix}$$

Figure 2a depicts the production network \mathbf{T} and Figure 2b depicts the similarity network, $\mathbf{T}\mathbf{T}^\top$. Input 1 is used only in the production of final good 1, while input 2 is used only in the production of final good 2. As a result, $D_{12} = 0$. Input 3 is used for the production of both final goods. Hence, inputs 1 and 3 (and 2 and 3) are strict strategic substitutes (i.e., $D_{13} > 0$ and $D_{23} > 0$).

When final demands are independent, $\mathbf{D} = \mathbf{T}\mathbf{T}^\top$. The spectral decomposition of this matrix \mathbf{D} is

$$\mathbf{D} = \mathbf{U}\mathbf{\Sigma}\mathbf{U}^\top, \quad \text{where } \mathbf{\Sigma} = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad \text{and} \quad \mathbf{U} = \begin{bmatrix} \frac{1}{2} & \frac{1}{\sqrt{2}} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{\sqrt{2}} & \frac{1}{2} \\ \frac{1}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{2}} \end{bmatrix}.$$

In this case, Proposition 1 tells us that the pass-through of the first eigenbundle \mathbf{u}^1 is $\lambda_1 = \frac{1}{3}$ while the pass-through of the second eigenbundle \mathbf{u}^2 is $\lambda_2 = \frac{1}{2}$. Hence, by Proposition 2, in order to maximize consumer welfare, the optimal policy taxes the first eigenbundle—the one with relatively low pass through—in order to subsidize the others.

To make this concrete, assume that final demands are given by: $Q_1 = 1$ and $Q_2 = \alpha \in [0, 1]$. Using the matrix \mathbf{T} , we then have the demands for inputs are

$$q_1 = 1, \quad q_2 = \alpha, \quad q_3 = \frac{1 + \alpha}{\sqrt{2}}.$$

Using $\underline{q}_\ell = \sum_i u_i^\ell q_i$ yields these quantities in the basis of eigenbundles:

$$\begin{aligned} \underline{q}_1 &= \frac{1}{2} + \frac{1}{2}\alpha + \frac{1+\alpha}{2} = 1 + \alpha \\ \underline{q}_2 &= \frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}}\alpha = \frac{1}{\sqrt{2}}(1 - \alpha) \\ \underline{q}_3 &= \frac{1}{2} + \frac{1}{2}\alpha - \frac{1+\alpha}{2} = 0 \end{aligned}$$

We next turn to optimal taxes and subsidies. When $\alpha = 1$, both \underline{q}_2 and \underline{q}_3 are zero. In this case, the planner cannot improve consumer welfare, because the system is effectively one-dimensional.

Assuming instead that $\alpha < 1$, as $\underline{q}_3 = 0$, it follows from Proposition 2, that $\underline{q}_1 \tau_1 > 0$ and $\underline{q}_2 \tau_2 < 0$. Since $\underline{q}_1 > 0$ and $\underline{q}_2 > 0$, it follows that $\tau_1 > 0$ and $\tau_2 < 0$. Using that $\tau_i = \sum_\ell u_i^\ell \tau_\ell$, and that $\tau_3 = 0$, we see that

$$\tau_1 = \frac{1}{2}\tau_1 + \frac{1}{\sqrt{2}}\tau_2, \quad \tau_2 = \frac{1}{2}\tau_1 - \frac{1}{\sqrt{2}}\tau_2 \quad \text{and} \quad \tau_3 = \frac{1}{\sqrt{2}}\tau_1.$$

The optimal policy taxes suppliers 2 and 3 and subsidizes supplier 1. Note that taxing supplier 3 not only raises tax money to be able to subsidize the other two suppliers, but it also achieves an indirect reduction of the prices of both suppliers 1 and 2 (via the strategic substitutability between supplier 3 and the other two suppliers).

This example illustrates how the nature of pass-throughs interacts with the size of suppliers to determine the optimal taxes and subsidies. On the one hand, note that when $\alpha = 1$, there is heterogeneity in both pass-through and in quantities, but their

interaction implies that the planner cannot improve welfare using a budget-neutral policy. On the other hand, when $\alpha < 1$, suppliers 1 and 2 are symmetric in terms of pass-through, but the optimal policy of subsidizing 1 and taxing 3 does increase consumer welfare.

6. CONCLUSION

The paper studies firms interacting strategically in an environment with rich heterogeneity. We bring a network perspective to the interactions between the different firms. The key contribution is to decompose any cost change into components that have a very convenient form: for these components, there is an unambiguous ranking of pass-throughs from cost changes to equilibrium prices. As an application, we show how a policymaker can use this ranking to design a tax policy that maximizes consumer welfare: The optimal policy leverages the strategic interactions among producers (in particular, that taxes on some dimensions have a higher pass-through to equilibrium prices) to tax some producers in order to use the tax revenue to subsidize other producers. The fact that the deadweight loss from taxation increases as a quadratic function of its size restricts the size of this tax program and its associated consumer surplus gains.

In this paper we have worked with linear demands for simplicity. In ongoing work, we extend our analysis to more general settings. The key idea is to consider small tax changes around a given status quo. This builds on a large literature on the so-called “tax reform approach” initiated by [Feldstein \(1976\)](#) and [Diewert \(1978\)](#), [Dixit \(1979\)](#) and [Tirole and Guesnerie \(1981\)](#). We offer a new foundation for the tax reform approach: reform is implemented with some noise and the policymaker is averse to variance in the outcome. In this analysis, we derive a very similar characterization for the optimal tax/subsidy policy in terms of the spectral decomposition of demand spillovers. The analysis of small policy changes actually makes many of the characterizations simpler.

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APPENDIX A. PROOFS

A.1. A canonical market. A key technical step is to transform an arbitrary environment satisfying our assumptions into an isomorphic canonical form, that of [subsubsection 2.3.2](#).

Start with a general market satisfying our assumptions, where

$$U(\mathbf{Q}) = \sum_f \beta_f Q_f - \frac{1}{2} \mathbf{Q}^\top \mathbf{B} \mathbf{Q}$$

is the utility over final goods. This is associated with a Jacobian matrix Δ such that

$$\Delta_{ff'} = \partial Q_f / \partial P_{f'},$$

and $\Delta = \mathbf{B}^{-1}$, where \mathbf{B} is positive definite. In addition, there is a supply network \mathbf{T} . Let $\hat{\mathbf{D}}$ be defined by $\hat{D}_{ij} = \partial q_i(\mathbf{p}) / \partial p_i$. We can calculate that

$$\hat{\mathbf{D}} = \mathbf{T} \Delta \mathbf{T}^\top.$$

Let $\text{Diag}(\hat{\mathbf{D}})$ be $\hat{\mathbf{D}}$ with all off-diagonal entries set to zero. Then we can see from the definition of \mathbf{D} that

$$\mathbf{D} = \text{Diag}(\hat{\mathbf{D}})^{-1} \hat{\mathbf{D}}.$$

The assumption that \mathbf{D} is positive semidefinite, and in particular symmetric, implies that the diagonal entries of $\hat{\mathbf{D}}$ are all equal; we may (by rescaling units) take them all to be -1 . Then

$$\mathbf{D} = -\mathbf{T} \Delta \mathbf{T}^\top,$$

which is positive definite under our assumptions.

Consumer welfare when the consumer makes her optimal demand given prices \mathbf{P} is

$$U^* = \frac{1}{2} \mathbf{Q}(\mathbf{P})^\top \Delta^{-1} \mathbf{Q}(\mathbf{P}).$$

Now, we will introduce a reparameterization which we will denote by tildes. Because \mathbf{B} is positive definite, so is its inverse Δ , and Δ therefore has a positive definite square root, which we write $\Delta^{1/2}$. Let

$$\tilde{\mathbf{Q}} = \Delta^{-1/2} \mathbf{Q} \quad \text{and} \quad \tilde{\mathbf{T}} = \mathbf{T} \Delta^{1/2}.$$

Then

$$\tilde{U}^* = U^* = \frac{1}{2} \tilde{\mathbf{Q}}^\top \tilde{\mathbf{Q}}$$

and

$$\mathbf{D} = \tilde{\mathbf{D}} = \tilde{\mathbf{T}}^\top \tilde{\mathbf{T}}.$$

The reparameterization makes the final goods independent, and all the interactions in the consumption of final goods are encoded in $\tilde{\mathbf{T}}$.

This calculation shows that, under our standing assumption that D is a positive definite matrix, a linear transformation of final goods allows us to assume that final goods have independent, unit-slope demands ($\Delta = -I$) without changing either the game among suppliers of intermediates (given by D) or the consumer surplus function when the consumer best-responds (U^*). We will therefore work with an economy satisfying these assumptions throughout the proofs, which permits an exposition with some convenient geometric interpretations.

A.2. Some useful definitions and identities. Throughout this section, we repeatedly use the fact that eigenbundles are orthonormal. Moreover, we will use the singular vector basis for the final goods space (recall [subsection 3.1](#)). For any $\mathbf{X} \in \mathbb{R}^F$, let $\underline{\mathbf{X}}$ denote $\mathbf{V}^\top \mathbf{X}$; that is, $\underline{\mathbf{X}}$ is the profile \mathbf{X} expressed in the basis \mathbf{V} .

We have that

$$\mathbf{T} = \sum_{\ell} s_{\ell} \mathbf{u}^{\ell} (\mathbf{v}^{\ell})^\top \text{ and } \mathbf{T}^\top = \sum_{\ell} s_{\ell} \mathbf{v}^{\ell} (\mathbf{u}^{\ell})^\top,$$

and that

$$\mathbf{T}\mathbf{T}^\top = \sum_{\ell, m} s_{\ell} \mathbf{u}^{\ell} \mathbf{v}^{\ell \top} s_m \mathbf{v}^m (\mathbf{u}^m)^\top = \sum_{\ell} s_{\ell}^2 \mathbf{u}^{\ell} (\mathbf{u}^{\ell})^\top.$$

Furthermore, define $\mathbf{M} := (\mathbf{I} + \mathbf{T}\mathbf{T}^\top)^{-1}$ and note that

$$\mathbf{M} = \sum_{\ell} \frac{1}{1 + s_{\ell}^2} \mathbf{u}^{\ell} (\mathbf{u}^{\ell})^\top.$$

Definition 1. Let $\boldsymbol{\tau} = \sum_{\ell} x_{\ell} \mathbf{u}^{\ell}$; that is, x_{ℓ} denotes the tax on the eigenbundle \mathbf{u}^{ℓ} .

We have that $\mathbf{M}\boldsymbol{\tau} = \sum_{\ell} \frac{1}{1 + s_{\ell}^2} \mathbf{u}^{\ell} (\mathbf{u}^{\ell})^\top \sum_m x_m \mathbf{u}^m = \sum_{\ell} \frac{1}{1 + s_{\ell}^2} x_{\ell} \mathbf{u}^{\ell}$, so

$$\mathbf{D}\mathbf{M}\boldsymbol{\tau} = \sum_{\ell} \frac{\sigma_{\ell}}{1 + s_{\ell}^2} x_{\ell} \mathbf{u}^{\ell}$$

Definition 2. Let $\mathbf{Q} = \sum_{\ell} \underline{Q}_{\ell} \mathbf{v}^{\ell}$ and $\mathbf{q} = \sum_{\ell} \underline{q}_{\ell} \mathbf{u}^{\ell}$, so that \underline{Q}_{ℓ} and \underline{q}_{ℓ} denote the amount of bundle \mathbf{v}^{ℓ} and \mathbf{u}^{ℓ} produced, respectively.

Note that $\mathbf{q} = \mathbf{T}\mathbf{Q} = \sum_{\ell} s_{\ell} \underline{Q}_{\ell} \mathbf{u}^{\ell}$, so $\underline{q}_{\ell} = s_{\ell} \underline{Q}_{\ell}$. We may choose the orientations (multiplying them by -1 if necessary) of the eigenbundles so that $\underline{q}_{\ell} \geq 0$, and hence also that $\underline{Q}_{\ell} \geq 0$, for all ℓ .

A.3. Proofs of the main results. We are now ready to analyze the optimization problem at the center of our results.

A.3.1. Rewriting the optimal tax problem.

Lemma 2. We can rewrite the optimal tax problem as

$$\begin{aligned} & \underset{d_{\ell}}{\text{Maximize}} && \sum_{\ell} (d_{\ell} + \underline{Q}_{\ell})^2 \\ & \text{subject to} && \sum_{\ell} \frac{(d_{\ell} + \underline{Q}_{\ell}/2)^2}{K\lambda_{\ell}} = 1, \end{aligned} \tag{13}$$

where $d_{\ell} := -x_{\ell} \frac{s_{\ell}}{1 + s_{\ell}^2}$ is the change in the amount of the quantity of the bundle \mathbf{v}^{ℓ} induced by the optimal policy.

Remark 1. Note that the optimization problem (13) amounts to choosing the quantity profile that is furthest from the origin on the ellipsoid with center $\mathbf{Q}/2$ and whose axis ℓ is in the direction \mathbf{v}^ℓ and has length $\sqrt{K\lambda_\ell}$.

Proof of Lemma 2. The consumer surplus associated with final good f is $\frac{1}{2}Q_f^2$, so $2CS = \sum_f Q_f^2$. Hence, the increase ΔCS in consumer surplus is proportional to $2\langle \mathbf{Q}, \Delta \mathbf{Q} \rangle + \langle \Delta \mathbf{Q}, \Delta \mathbf{Q} \rangle$, where \mathbf{Q} denotes the initial profile of final good quantities (with $\tau = \mathbf{0}$), and $\Delta \mathbf{Q}$ denotes the change in \mathbf{Q} induced by τ . Since $\mathbf{Q} := \mathbf{1} - \mathbf{P}$ and $\Delta \mathbf{P} = \mathbf{T}^\top \mathbf{M} \tau$, we have that $\Delta \mathbf{Q} := -\mathbf{T}^\top \mathbf{M} \tau$. Hence, our objective is to maximize

$$-2\langle \mathbf{Q}, \mathbf{T}^\top \mathbf{M} \tau \rangle + \langle \mathbf{T}^\top \mathbf{M} \tau, \mathbf{T}^\top \mathbf{M} \tau \rangle$$

The budget-balance condition is $\sum_i \tau_i q_i = 0$, or $\langle \mathbf{q}, \tau \rangle = 0$, which, using that $\mathbf{q} = \mathbf{T} \mathbf{Q}$ and that $\mathbf{Q} = \mathbf{1} - \mathbf{T}^\top \mathbf{p}$, we can write as $\langle \mathbf{T}(\mathbf{1} - \mathbf{T}^\top \mathbf{p}), \tau \rangle = 0$, or $\langle \mathbf{T} \mathbf{Q} - \mathbf{T} \mathbf{T}^\top \mathbf{M} \tau, \tau \rangle = 0$. Hence, our problem is

$$\text{Maximize } -2\langle \mathbf{Q}, \mathbf{T}^\top \mathbf{M} \tau \rangle + \langle \mathbf{T}^\top \mathbf{M} \tau, \mathbf{T}^\top \mathbf{M} \tau \rangle \text{ s.t. } \langle \mathbf{T} \mathbf{Q}, \tau \rangle - \langle \mathbf{T} \mathbf{T}^\top \mathbf{M} \tau, \tau \rangle = 0, \quad (14)$$

which, using our definitions above, we can rewrite as

$$\begin{aligned} \text{Maximize } & -2 \left\langle \sum_\ell \underline{Q}_\ell \mathbf{v}^\ell, \sum_\ell \frac{s_\ell}{1+s_\ell^2} x_\ell \mathbf{v}^\ell \right\rangle + \left\langle \sum_\ell \frac{s_\ell}{1+s_\ell^2} x_\ell \mathbf{v}^\ell, \sum_\ell \frac{s_\ell}{1+s_\ell^2} x_\ell \mathbf{v}^\ell \right\rangle \\ \text{subject to } & \left\langle \sum_\ell s_\ell \underline{Q}_\ell \mathbf{u}^\ell, \sum_\ell x_\ell \mathbf{u}^\ell \right\rangle - \left\langle \sum_\ell \frac{s_\ell^2}{1+s_\ell^2} x_\ell \mathbf{u}^\ell, \sum_\ell x_\ell \mathbf{u}^\ell \right\rangle = 0 \end{aligned}$$

or, equivalently,

$$\begin{aligned} \text{Maximize } & \sum_\ell \left[-2\underline{Q}_\ell \frac{s_\ell}{1+s_\ell^2} x_\ell + \left(\frac{s_\ell}{1+s_\ell^2} \right)^2 x_\ell^2 \right] \\ \text{subject to } & \sum_\ell \left[\underline{Q}_\ell s_\ell x_\ell - \frac{s_\ell^2}{1+s_\ell^2} x_\ell^2 \right] = 0 \end{aligned}$$

which is equivalent to (13). \square

A.3.2. Optimal taxes as a function of Lagrange multiplier. The Lagrangian corresponding to (13) is

$$\mathcal{L} = \sum_\ell \left(d_\ell + \underline{Q}_\ell \right)^2 - z \left(\sum_\ell \frac{(d_\ell + \underline{Q}_\ell/2)^2}{K\lambda_\ell} - 1 \right). \quad (15)$$

The first-order condition with respect to d_ℓ gives

$$d_\ell + \underline{Q}_\ell = z \frac{d_\ell + \underline{Q}_\ell/2}{K\lambda_\ell}$$

or, assuming that $K\lambda_\ell \neq z$ (which we will verify below),

$$d_\ell = -\frac{\underline{Q}_\ell}{2} \frac{z - 2K\lambda_\ell}{z - K\lambda_\ell} \quad (16)$$

or, using that $d_\ell := -x_\ell \frac{s_\ell}{1+s_\ell^2}$,

$$x_\ell = \frac{1 + s_\ell^2}{s_\ell^2} \frac{Q_\ell}{2} \frac{z - 2K\lambda_\ell}{z - K\lambda_\ell} \quad (17)$$

Proposition 3 below says that, if $Q_\ell > 0$, then $0 < \frac{K\lambda_\ell}{z} < 1$, so (17) above implies that $x_\ell > 0$ as long as $\lambda_\ell < \frac{z}{2K}$, and $x_\ell < 0$ as long as $\lambda_\ell > \frac{z}{2K}$.

A.3.3. *Solving for the Lagrange multiplier.* Combining (16) with the budget constraint

$$\sum_\ell \frac{(d_\ell + \frac{Q_\ell}{2})^2}{K\lambda_\ell} = 1$$

and maintaining the assumption that $K\lambda_\ell \neq z$, we get that

$$K \sum_\ell \lambda_\ell \left(\frac{Q_\ell}{2} \frac{1}{z - K\lambda_\ell} \right)^2 = 1. \quad (18)$$

Proposition 3. Equation (18) has a strictly positive root z , satisfying, for all ℓ with $Q_\ell > 0$,

$$z > K\lambda_\ell.$$

Moreover, this is the unique real root.

Proof. The left-hand side of (18) asymptotes to infinity as $z \rightarrow K\lambda_\ell$ for each ℓ with $Q_\ell > 0$, asymptotes to 0 as $z \rightarrow \infty$, and is decreasing in z when $z > K\lambda_{\ell'}$, where ℓ' denotes the largest component ℓ with $Q_\ell > 0$. Hence, there exists a strictly positive root z of (18), satisfying, for all ℓ with $Q_\ell > 0$, $z > K\lambda_\ell$. Uniqueness follows from the fact that, by (16), d_ℓ is strictly decreasing in z when $Q_\ell > 0$, and that d_ℓ is uniquely pinned down, since there is a unique point on the ellipsoid described in Remark 1 that is furthest from the origin. \square