

# THOMPSON'S GROUP $F$ IS NOT STRONGLY AMENABLE

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ABSTRACT. We show that Thompson's group  $F$  has a topological action on a compact metric space that is proximal and has no fixed points.

## 1. INTRODUCTION

In his book “Proximal Flows” [9, Section II.3, p. 19] Glasner defines the notion of a *strongly amenable group*: A group is strongly amenable if each of its proximal actions on a compact space has a fixed point. A continuous action  $G \curvearrowright X$  of a topological group on a compact Hausdorff space is proximal if for every  $x, y \in X$  there exists a net  $\{g_n\}$  of elements of  $G$  such that  $\lim_n g_n x = \lim_n g_n y$ .

Glasner shows that virtually nilpotent groups are strongly amenable and that non-amenable groups are not strongly amenable. He also gives examples of amenable — in fact, solvable — groups that are not strongly amenable. Glasner and Weiss [7] construct proximal minimal actions of the group of permutations of the integers, and Glasner constructs proximal flows of Lie groups [8]. To the best of our knowledge there are no other such examples known. Furthermore, there are no other known examples of minimal proximal actions that are not also *strongly proximal*. An action  $G \curvearrowright X$  is strongly proximal if the orbit closure of every Borel probability measure on  $G$  contains a point mass measure. This notion, as well as that of the related Furstenberg boundary [4–6], have been the object of a much larger research effort, in particular because a group is amenable if and only if all of its strongly proximal actions on compact spaces have fixed points.

Richard Thompson's group  $F$  has been alternatively “proved” to be amenable and non-amenable (see, e.g., [2]), and the question of its amenability is currently unresolved. In this paper we pursue the less ambitious goal of showing that it is not strongly amenable, and do so by directly constructing a proximal action that has no fixed points.

This action does admit an invariant measure, and thus does not provide any information about the amenability of  $F$ . It is a new example of a proximal action which is not strongly proximal.

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## 2. PROOFS

Let  $F$  denote Thompson's group  $F$ . In the representation of  $F$  as a group of piecewise linear transformations of  $\mathbb{R}$  (see, e.g., [10, Section 2.C]), it is generated by  $a$  and  $b$  which are given by

$$a(x) = x - 1$$

$$b(x) = \begin{cases} x & x \leq 0 \\ x/2 & 0 \leq x \leq 2 \\ x - 1 & 2 \leq x. \end{cases}$$

The set of dyadic rationals  $\Gamma = \mathbb{Z}[\frac{1}{2}]$  is the orbit of 0. The Schreier graph of the action  $G \curvearrowright \Gamma$  with respect to the generating set  $\{a, b\}$  is shown in Figure 1 (see [10, Section 5.A, Figure 6]). The solid lines denote the  $a$  action and the dotted lines denote the  $b$  action; self-loops (i.e., points stabilized by a generator) are omitted. This graph consists of a tree-like structure (the blue and white nodes) with infinite chains attached to each node (the red nodes).

Equipped with the product topology,  $\{-1, 1\}^\Gamma$  is a compact space on which  $F$  acts continuously by shifts:

$$(2.1) \quad [fx](\gamma) = x(f^{-1}\gamma).$$

**Proposition 2.1.** *Let  $c_{-1}, c_{+1} \in \{-1, 1\}^\Gamma$  be the constant functions. Then for any  $x \in \{-1, 1\}^\Gamma$  it holds that at least one of  $c_{-1}, c_{+1}$  is in the orbit closure  $\overline{Fx}$ .*

*Proof.* It is known that the action  $F \curvearrowright \Gamma$  is highly-transitive (Lemma 4.2 in [3]), i.e. for every finite  $V, W \subset \Gamma$  of the same size there exists a  $f \in F$  such that  $f(V) = W$ . Let  $x \in \{-1, 1\}^\Gamma$ . There is at least one of -1 and 1, say  $\alpha$ , for which we have infinitely many  $\gamma \in \Gamma$  with  $x(\gamma) = \alpha$ . Given a finite  $W \subset \Gamma$  choose a  $V \subset \Gamma$  of the same size and such that  $x(\gamma) = \alpha$  for all  $\gamma \in V$ . Then there is some  $f \in F$  with  $f(V) = W$ , and so  $fx$  takes the value  $\alpha$  on  $W$ . Since  $W$  is arbitrary we have that  $c_\alpha$  is in the orbit closure of  $x$ .  $\square$



$\{0, 1\}^F$  of the subgroups of  $F$ , the subgroup  $K$  is the limit of the subgroups  $K_n$ , where  $K_n$  is the stabilizer of  $1/2^n$ . It is easy to verify that  $K$  is the subgroup of  $F$  consisting of the transformations that stabilize 0 and have right derivative 1 at 0 (although this fact will not be important). Let  $\Lambda = F/K$ .

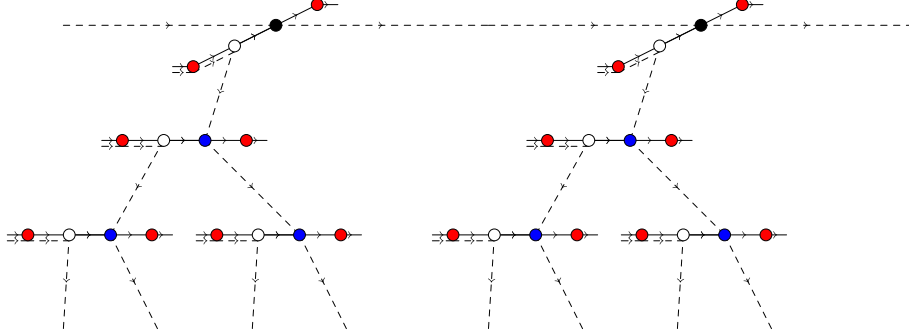


FIGURE 2. The action of  $F$  on  $\Lambda$ .

We can naturally identify with  $\mathbb{Z}$  the chain black nodes at the top of  $\Lambda$  (see Figure 2). Let  $\Lambda'$  be the subgraph of  $\Lambda$  in which the dotted edges connecting the black nodes have been removed. Given a black node  $n \in \mathbb{Z}$ , denote by  $T_n$  the connected component of  $n$  in  $\Lambda'$ ; this includes the black node  $n$ , the chain that can be reached from it using solid edges, and the entire tree that hangs from it. Each graph  $T_n$  is isomorphic to the Schreier graph of  $\Gamma$ , and so the graph  $\Lambda$  is a covering graph of  $\Gamma$  (in the category of Schreier graphs). Let

$$\Psi: \Lambda \rightarrow \Gamma$$

be the covering map. That is,  $\Psi$  is a graph isomorphism when restricted to each  $T_n$ , with the black nodes in  $\Lambda$  mapped to the black node  $0 \in \Gamma$ .

Using the map  $\Psi$  we give names to the nodes in  $\Lambda$ . Denote the nodes in  $T_0$  as  $\{(0, \gamma) : \gamma \in \Gamma\}$  so that  $\Psi(0, \gamma) = \gamma$ . Likewise, in each  $T_n$  denote by  $(n, \gamma)$  the unique node in  $T_n$  that  $\Psi$  maps to  $\gamma$ . Hence we identify  $\Lambda$  with

$$\mathbb{Z} \times \Gamma = \{(n, \gamma) : n \in \mathbb{Z}, \gamma \in \Gamma\}$$

and the  $F$ -action is given by

$$(2.2) \quad a(n, \gamma) = (n, a\gamma)$$

$$(2.3) \quad b(n, \gamma) = \begin{cases} (n, b\gamma) & \text{if } \gamma \neq 0 \\ (n + 1, 0) & \text{if } \gamma = 0 \end{cases}$$

Equip  $\{-1, 1\}^\Lambda$  with the product topology to get a compact space. As usual, the  $F$ -action on  $\Lambda$  (given explicitly in 2.2 and 2.3) defines a continuous action on  $\{-1, 1\}^\Lambda$ .

Consider  $\pi : \{-1, 1\}^\Gamma \rightarrow \{-1, 1\}^\Lambda$ , given by  $\pi(x)(n, \gamma) = (-1)^n x(\gamma)$ . Let  $Y = \pi(\{-1, 1\}^\Gamma) \subseteq \{-1, 1\}^\Lambda$ .

**Claim 2.2.**  *$Y$  is compact and  $F$ -invariant.*

*Proof.*  $\pi$  is injective and continuous, so  $Y = \pi(\{-1, 1\}^\Gamma) \subseteq \{-1, 1\}^\Lambda$  is compact and isomorphic to  $\{-1, 1\}^\Gamma$ . Moreover,  $Y$  is invariant to the action of  $F$ , because  $a^{\pm 1}\pi(x) = \pi(a^{\pm 1}x)$  and  $b^{\pm 1}\pi(x) = \pi(b^{\pm 1}\bar{x})$  where  $\bar{x}(\gamma) = \begin{cases} x(\gamma) & \text{if } \gamma \neq 0 \\ -x(\gamma) & \text{if } \gamma = 0 \end{cases}$ .  $\square$

The last  $F$ -space we define is  $Z$ , the set of pairs of mirror image configurations in  $Y$ :

$$(2.4) \quad Z = \{\{y, -y\} : y \in Y\}.$$

Now it is clear that equipped with the quotient topology,  $Z$  is a compact and Hausdorff  $F$ -space. Furthermore, we now observe that  $Z$  admits an invariant measure. Consider the i.i.d. Bernoulli 1/2 measure on  $\{-1, 1\}^\Gamma$ , i.e. the unique Borel measure on  $\{-1, 1\}^\Gamma$ , for which

$$X_\gamma : \{-1, 1\}^\Gamma \rightarrow \{0, 1\}, \quad x \mapsto \frac{x(\gamma) + 1}{2}$$

are independent Bernoulli 1/2 random variables for all  $\gamma \in \Gamma$ . Clearly, it is an invariant measure and hence it is pushed forward to an invariant measure on  $Y$ , and then on  $Z$ . In particular, this shows that  $Z$  is not strongly proximal.

**Claim 2.3.** *The action  $F \curvearrowright Z$  does not have any fixed points.*

*Proof.* Pick  $\hat{y} = \{y, -y\} \in Z$ . We have  $[by](0, -1) = y(0, -1) \neq -y(0, -1)$ , so  $by \neq -y$ . Similarly,  $[by](0, 0) = y(-1, 0) = -y(0, 0) \neq y(0, 0)$ , and so  $by \neq y$ . Hence  $b\hat{y} \neq \hat{y}$ .  $\square$

**Proposition 2.4.** *The action  $F \curvearrowright Z$  is proximal.*

*Proof.* Let  $\hat{y}_1 = \{y_1, -y_1\}$  and  $\hat{y}_2 = \{y_2, -y_2\}$  be two points in  $Z$ , and let  $y_i = \pi(x_i)$ .

Let  $x_1 \cdot x_2$  denote the pointwise product of  $x_1$  and  $x_2$ . Now by Proposition 2.1 there is a sequence of elements  $\{f_n\}_n$  in  $F$  such that  $\{f_n(x_1 \cdot x_2)\}_n$  tends to either  $c_{-1}$  or  $c_{+1}$  in  $\{-1, 1\}^\Gamma$ . Since  $Y$  is compact, we may assume that  $\{f_n y_1\}_n$  and  $\{f_n y_2\}_n$  have limits, by descending to a subsequence if necessary.

It is straightforward to check that  $f_n y_1 \cdot f_n y_2 = f_n \pi(x_1) \cdot f_n \pi(x_2) = \pi(f_n x_1) \cdot \pi(f_n x_2)$ . So:

$$\begin{aligned} [f_n y_1 \cdot f_n y_2](n, \gamma) &= [\pi(f_n x_1) \cdot \pi(f_n x_2)](n, \gamma) \\ &= (-1)^{2n} [f_n x_1](\gamma) [f_n x_2](\gamma) \\ &= [f_n x_1 \cdot f_n x_2](\gamma) = [f_n(x_1 \cdot x_2)](\gamma) \end{aligned}$$

So  $\lim_n f_n y_1 = \pm \lim_n f_n y_2$ , which implies  $\lim_n f_n \hat{y}_1 = \lim_n f_n \hat{y}_2$ .  $\square$

**Theorem 2.5.** *Thompson's group  $F$  is not strongly amenable.*

*Proof.* Since the space  $Z$  we constructed above is proximal (Proposition 2.4), and has no fixed points (Claim 2.3), we conclude that  $F$  has a proximal action with no fixed points, so  $F$  is not strongly amenable.  $\square$

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