MA144A, HOMEWORK 1
DUE THURSDAY, OCTOBER 11TH

Collaboration on homework is encouraged, but individually written solutions are required. Also, please name all collaborators and sources of information on each assignment; any such named source may be used.

(1) Let I be a set, and let $\{\mathcal{F}_i\}_{i\in I}$ be a collection of sigma-algebras of subsets of Ω. Show that $\cap_{i\in I}\mathcal{F}_i$ is a sigma-algebra.

(2) Let $\mathcal{C}$ be a collection of subsets of Ω. Show that there exists a unique minimal (under inclusion) sigma-algebra $\mathcal{F} \supseteq \mathcal{C}$.

(3) Consider $\mathcal{A}_{\text{clopen}}$, as defined in Example 2.5 in the lectur notes. Prove that there exists an additive $\mu_0\colon \mathcal{A}_{\text{clopen}} \to [0,1]$ with

$$\mu_0(A_x) = 2^{-|x|}.$$

Bonus: Prove that there exists a countably additive such μ_0, so that whenever $(A_1, A_2, \ldots)$ are disjoint elements of $\mathcal{A}_{\text{clopen}}$ with $\cup_n A_n \in \mathcal{A}_{\text{clopen}}$ then $\mu_0(\cup_n A_n) = \sum_n \mu_0(A_n)$.

(4) A *finitely additive probability measure* on an algebra $\mathcal{A}$ of subsets of Ω is a map $\mu\colon \mathcal{A} \to [0,1]$ such that $\mu(\Omega) = 1$ and such that if $A_1, A_2 \in \mathcal{A}$ are disjoint then $\mu(A_1 \cup A_2) = \mu(A_1) + \mu(A_2)$.

Let $\Omega = \mathbb{Z}$ and $\mathcal{F}$ be the power set of $\mathbb{Z}$. A finitely additive probability measure $\mu\colon \mathcal{F} \to [0,1]$ is *shift-invariant* if for all $A \in \mathcal{F}$ it holds that $\mu(A) = \mu(A+1)$, where

$$A + 1 = \{n + 1 \,:\, n \in A\}.$$

(a) Prove that if μ is shift-invariant then it is not sigma-additive.
(b) *Bonus.* Prove that there exists such a shift-invariant μ.

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