MA144A, Homework 1  
Due Thursday, October 11th

Collaboration on homework is encouraged, but individually written solutions are required. Also, please name all collaborators and sources of information on each assignment; any such named source may be used.

(1) Let $I$ be a set, and let $\{F_i\}_{i \in I}$ be a collection of sigma-algebras of subsets of $\Omega$. Show that $\bigcap_{i \in I} F_i$ is a sigma-algebra.

(2) Let $\mathcal{C}$ be a collection of subsets of $\Omega$. Show that there exists a unique minimal (under inclusion) sigma-algebra $\mathcal{F} \supseteq \mathcal{C}$.

(3) Consider $\mathcal{A}_{\text{clopen}}$, as defined in Example 2.5 in the lecture notes. Prove that there exists an additive $\mu_0 : \mathcal{A}_{\text{clopen}} \to [0, 1]$ with

$$\mu_0(A_x) = 2^{-|x|}.$$ 

*Bonus:* Prove that there exists a countably additive such $\mu_0$, so that whenever $(A_1, A_2, \ldots)$ are disjoint elements of $\mathcal{A}_{\text{clopen}}$ with $\cup_n A_n \in \mathcal{A}_{\text{clopen}}$ then $\mu_0(\cup_n A_n) = \sum_n \mu_0(A_n)$.

(4) A finitely additive probability measure on an algebra $\mathcal{A}$ of subsets of $\Omega$ is a map $\mu : \mathcal{A} \to [0, 1]$ such that $\mu(\Omega) = 1$ and such that if $A_1, A_2 \in \mathcal{A}$ are disjoint then $\mu(A_1 \cup A_2) = \mu(A_1) + \mu(A_2)$.

Let $\Omega = \mathbb{Z}$ and $\mathcal{F}$ be the power set of $\mathbb{Z}$. A finitely additive probability measure $\mu : \mathcal{F} \to [0, 1]$ is *shift-invariant* if for all $A \in \mathcal{F}$ it holds that $\mu(A) = \mu(A + 1)$, where $A + 1 = \{n + 1 : n \in A\}$.

(a) Prove that if $\mu$ is shift-invariant then it is not sigma-additive.

(b) *Bonus.* Prove that there exists such a shift-invariant $\mu$. 

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