Ma144a, Homework 2
Due Thursday, October 18th

Collaboration on homework is encouraged, but individually written solutions are required. Also, please name all collaborators and sources of information on each assignment; any such named source may be used.

(1) Consider the Bernoulli measure on $\{0,1\}^{\mathbb{N}}$ which is the unique extension (as we have shown in class / homework) of $\mu_0 \colon \mathcal{A}_{\text{clopen}} \to [0,1]$ with

$$\mu_0(A_x) = 2^{-|x|}.$$

(a) For $n \in \mathbb{N}$ let the real random variable $X_n \colon \Omega \to \mathbb{R}$ be given by $X_n(\omega) = \omega_n$. Show that X_n is indeed a random variable, and prove that the random variables $(X_1, X_2, \ldots)$ are independent.

(b) Let the real random variable $Y \colon \Omega \to \mathbb{R}$ be given by

$$Y(\omega) = \sum_{n=1}^{\infty} 2^{-n}\omega_n.$$

Prove that Y is indeed a random variable, and that its law is the uniform (Lebesgue) measure on $[0,1)$. (Hint: you can use the fact that the algebra of diadic intervals $[m2^{-n}, (m+1)2^{-n})$, $m < 2^n$ generates the Borel sigma-algebra on $[0,1)$.)

(c) Construct independent random variables $(Y_1, Y_2, \ldots)$, each with the uniform distribution on $[0,1)$.

(2) Consider a casino in which there is an infinite sequence of slot machines. On machine n a gambler gains a dollar with probability $1 - 2^{-n}$, and loses 2^n dollars with probability 2^{-n}. Consider a gambler who starts out with 0 dollars in her account, and proceeds to gamble on each machine in turn. That is, her balance is the sequence of random variables $\{X_n\}$ with $X_0 = 0$ and $X_{n+1} = X_n + Y_{n+1}$, where $\{Y_n\}$ is a sequence of independent random variables with $\mathbb{P}[Y_n = 1] = 1 - 2^{-n}$ and $\mathbb{P}[Y_n = -2^n] = 2^{-n}$.

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(a) Show that $\mathbb{E}[X_n] < 0$ for all $n > 0$.

(b) Show that $\mathbb{P}[\lim X_n = \infty] = 1$.