MA144A, HOMEWORK 2
DUE THURSDAY, OCTOBER 18th

Collaboration on homework is encouraged, but individually written solutions are required. Also, please name all collaborators and sources of information on each assignment; any such named source may be used.

(1) Consider the Bernoulli measure on \( \{0, 1\}^\mathbb{N} \) which is the unique extension (as we have shown in class / homework) of \( \mu_0: \mathcal{A}_{\text{clopen}} \to [0, 1] \) with

\[
\mu_0(A_x) = 2^{-|x|}.
\]

(a) For \( n \in \mathbb{N} \) let the real random variable \( X_n: \Omega \to \mathbb{R} \) be given by \( X_n(\omega) = \omega_n \). Show that \( X_n \) is indeed a random variable, and prove that the random variables \( (X_1, X_2, \ldots) \) are independent.

(b) Let the real random variable \( Y: \Omega \to \mathbb{R} \) be given by

\[
Y(\omega) = \sum_{n=1}^{\infty} 2^{-n} \omega_n.
\]

Prove that \( Y \) is indeed a random variable, and that its law is the uniform (Lebesgue) measure on \( [0, 1) \). (Hint: you can use the fact that the algebra of diadic intervals \([m2^{-n}, (m+1)2^{-n})\), \( m < 2^n \) generates the Borel sigma-algebra on \([0, 1)\).)

(c) Construct independent random variables \( (Y_1, Y_2, \ldots) \), each with the uniform distribution on \([0, 1)\).

(2) Consider a casino in which there is an infinite sequence of slot machines. On machine \( n \) a gambler gains a dollar with probability \( 1 - 2^{-n} \), and loses \( 2^n \) dollars with probability \( 2^{-n} \). Consider a gambler who starts out with 0 dollars in her account, and proceeds to gamble on each machine in turn. That is, her balance is the sequence of random variables \( \{X_n\} \) with \( X_0 = 0 \) and \( X_{n+1} = X_n + Y_{n+1} \), where \( \{Y_n\} \) is a sequence of independent random variables with \( \mathbb{P}[Y_n = 1] = 1 - 2^{-n} \) and \( \mathbb{P}[Y_n = -2^n] = 2^{-n} \).

Omer Tamuz. Email: tamuz@caltech.edu.
(a) Show that $E[X_n] < 0$ for all $n > 0$.

(b) Show that $P[\lim X_n = \infty] = 1$. 