MA140A, Homework 3
Due Friday, October 23rd

Collaboration on homework is encouraged, but individually written solutions are required. Also, please name all collaborators and sources of information on each assignment; any such named source may be used.

(1) Consider the sequence of independent random variables $X_1, X_2, \ldots$, where the cumulative distribution function of $X_n$ is given by

$$
P[X_n \leq x] = \begin{cases} 
2^{-n} \exp(2^{-n}x) & \text{if } x < 0 \\
2^{-n} & \text{if } 0 \leq x < 0.01 \\
1 & \text{if } 0.01 \leq x.
\end{cases}
$$

Consider a magical casino in which there is an infinite sequence of slot machines. A gambler starts out with 0 dollars in her account, and proceeds to gamble on each machine in turn. When she gambles on the $n^{th}$ machine her total yield is $X_n$. Hence her balance at time $n$ is $S_n$, where $S_0 = 0$ and

$$S_{n+1} = S_n + X_{n+1} = X_1 + \cdots + X_{n+1}.$$

(a) Prove that if a random variable $Y$ has a cumulative distribution function $F$, and if $F(x_0) = \lim_{x \to x_0} F(x) = p$, then $P[Y = x_0] = p$. Conclude that $P[X_n = 0.01] = 1 - 2^{-n}$.

(b) Show that $E[S_n] < 0$ for all $n > 0$. Use the definition of expectation and its properties as given in the lecture notes (as opposed to theorems not taught in this course).

(c) Show that $P[\lim S_n = \infty] = 1$.

(2) Prove the Dominated Convergence Theorem using the Monotone Convergence Theorem.

(3) Prove that there exists a simply normal number: a real number $x \in [0, 1]$ such that for any $d > 1$ and $a \in \{0, \ldots, d-1\}$ the digit $a$ occurs in the base $d$ representation of $x$ with asymptotic

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frequency $1/d$:

$$\lim_{n \to \infty} \frac{\text{number of times } a \text{ occurs in the first } n \text{ digits of } x}{n} = \frac{1}{d}.$$