MA144A, HOMEWORK 5
DUE FRIDAY, NOVEMBER 6^{TH}

Collaboration on homework is encouraged, but individually written solutions are required. Also, please name all collaborators and sources of information on each assignment; any such named source may be used.

(1) The *independent random walk* on $\mathbb{Z}^d$ is given by

$$P(x, y) = \begin{cases} 2^{-d} & \text{if } |x_i - y_i| = 1 \text{ for all } i \in \{1, \dots, d\} \\ 0 & \text{otherwise.} \end{cases}$$

Equivalently, the independent random walk on $\mathbb{Z}^d$ is a Markov process taking values on $\mathbb{Z}^d$ where in each coordinate the process is independent of the other coordinates, and equal to a simple random walk. For which d is the independent random walk transient, and for which is it recurrent?

(2) Let $\mathcal{T}_d$ be the set of nodes of the undirected d-regular tree. This is the unique undirected graph with no cycles and in which each node has degree d. Figure 1 shows $\mathcal{T}_4$.

The simple random walk on $\mathcal{T}_d$ is the Markov chain $(X_0, X_1, \dots)$ that starts from some $X_0 = s_0 \in \mathcal{T}_d$ and, at each step, transitions to each of the d neighboring nodes with probability $1/d$.

(a) Prove that for $d \geq 3$, the simple random walk on $\mathcal{T}_d$ is transient.

(b) Prove that for $d \geq 3$, the simple random walk on $\mathcal{T}_d$ has a non-trivial tail σ-algebra. Recall that this means that there is some event A that is in $\sigma(X_n, X_{n+1}, X_{n+2}, \dots)$ for every n, and such that $\mathbb{P}[A] \in (0, 1)$.

(3) Let P be a transition matrix over a countable state space S. A probability measure μ on S is said to be P-stationary if for all $x \in S$

$$\sum_{y \in S} P(y, x)\mu(y) = \mu(x).$$

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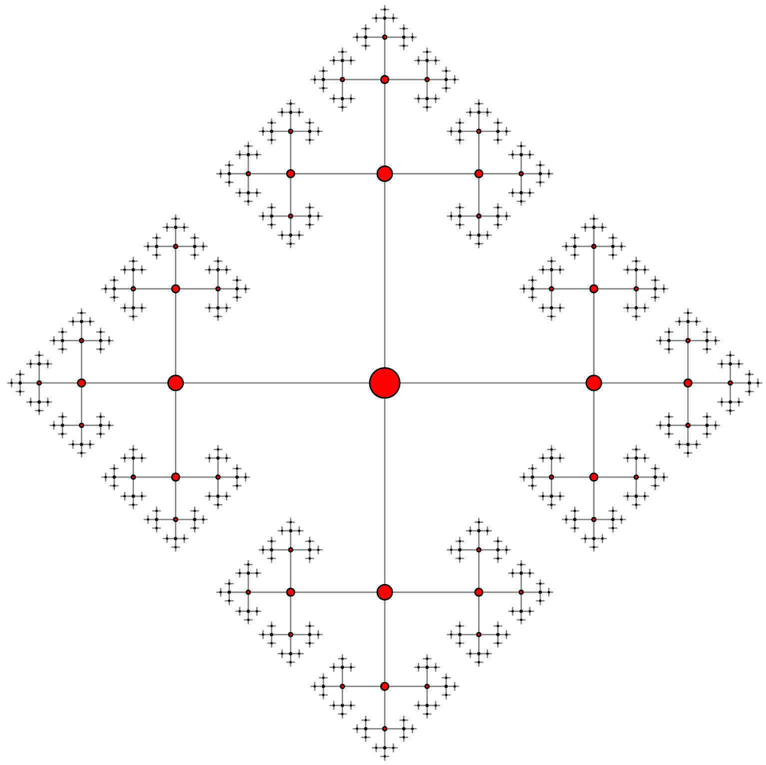

FIGURE 1. $\mathcal{T}_4$.

(a) Prove that if μ is P-stationary, X_0 has distribution μ, and $(X_0, X_1, \ldots)$ is a Markov chain with transition matrix P, then each X_n has distribution μ.

(b) Prove that if P is irreducible and transient then it has no stationary probability measures.

(c) Consider the transition matrix P on the state space $\{0, 1, 2, \ldots\}$ given by

$$P(i,j) = \begin{cases} 0 & \text{if } |i-j| > 1 \\ 2/3 & \text{if } j = i-1 \text{ and } i > 0 \\ 1/3 & \text{if } j = i+1 \text{ and } i > 0 \\ 1 & \text{if } j = 1 \text{ and } i = 0. \end{cases}$$

Prove that there exists a P-stationary probability measure.