MA144A, HOMEWORK 5
DUE MONDAY, NOVEMBER 12TH

Collaboration on homework is encouraged, but individually written solutions are required. Also, please name all collaborators and sources of information on each assignment; any such named source may be used.

(1) Let R_n be the fraction of black balls in Polya's urn. Show that $\lim_n R_n$ is distributed uniformly on $(0,1)$. Hint: calculate the distribution of R_n.

(2) Let P be a transition matrix over a countable state space S. A probability measure μ on S is said to be P-stationary if for all $x \in S$

$$\sum_{y \in S} P(y,x)\mu(y) = \mu(x).$$

(a) Prove that if μ is P-stationary, X_0 has distribution μ, and $(X_0, X_1, \ldots)$ is a Markov chain with transition matrix P, then each X_n has distribution μ.

(b) Prove that if P is irreducible and transient then it has no stationary probability measures.

(c) Prove that the simple random walk on $\mathbb{Z}$ does not have a P-stationary probability measure.

(d) Consider the transition matrix P on the state space $\{0,1,2,\ldots\}$ given by

$$P(i,j) = \begin{cases} 0 & \text{if } |i-j| > 1 \\ 2/3 & \text{if } j = i-1 \text{ and } i > 0 \\ 1/3 & \text{if } j = i+1 \text{ and } i > 0 \\ 1 & \text{if } j = 1 \text{ and } i = 0. \end{cases}$$

Prove that there exists a P-stationary probability measure.

(3) Prove Fatou's Lemma, as stated in the lecture notes, using the monotone convergence theorem. Hint: define $X_n(\omega) = \inf_{k \geq n} Z_k(\omega)$.

Omer Tamuz. Email: tamuz@caltech.edu.