MA144A, HOMEWORK 6
DUE MONDAY, NOVEMBER 19TH

Collaboration on homework is encouraged, but individually written solutions are required. Also, please name all collaborators and sources of information on each assignment; any such named source may be used.

(1) Let G be a group, let $(X_1, X_2, \ldots)$ be i.i.d. random variables taking values in G. Let $Z_n = X_1 \cdot X_2 \cdots X_n$ be a random walk on G.

Show if G is finite, and if the support of X_1 generates G, then the uniform distribution on G is the unique stationary distribution of this Markov chain.

Recall that a probability measure on the state space S of a Markov chain with transition matrix P is said to be stationary if for all $x \in S$

$$\mu(x) = \sum_{y \in S} \mu(y) P(y, x).$$

(2) Let $G = \langle a, b \rangle$ be the free group with two generators. Let $(X_1, X_2, \ldots)$ be i.i.d. and distributed uniformly on $\{a, b, a^{-1}, b^{-1}\}$, so that $Z_n = X_1 \cdots X_n$ is the simple random walk on G.

(a) Show that with probability 1, the length of the word Z_n tends to infinity. Here the length of $g \in G$ is the length of its shortest representation as a product of the symbols in $\{a, b, a^{-1}, b^{-1}\}$.

(b) Show that the Choquet-Deny Theorem does not apply to this (non-abelian) group: there is a bounded harmonic function, or, equivalently, the shift-invariant sigma-algebra is non-trivial.

(3) *Bonus.* Recall that a random variable is an equivalence class of measurable functions that are equal almost everywhere. A random variable is $\mathcal{G}$-measurable if in its equivalence class there is a $\mathcal{G}$-measurable function.

(a) Let P be the transition matrix of an irreducible Markov chain on a state space S such that $P(x, x) > \varepsilon$ for some

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$\varepsilon > 0$ and all $x \in S$. Show that a random variable is $\mathcal{T}$-measurable iff it is $\mathcal{I}$-measurable.

(b) Show that, for a random walk on a countable abelian group, a random variable is $\mathcal{I}$-measurable iff it is $\mathcal{T}$-measurable.

(4) *Kolmogorov from Choquet-Deny.* Let $(X_1, X_2, \ldots)$ be i.i.d. finitely supported random variables. Use the Choquet-Deny Theorem, together with 3b above, to show that their tail sigma-algebra is trivial.