Collaboration on homework is encouraged, but individually written solutions are required. Also, please name all collaborators and sources of information on each assignment; any such named source may be used.

(1) *The ultimatum game.* Consider the ultimatum game from the lecture notes.

(a) 20 points. How many possible strategy profiles does this game have?

(b) 20 points. Write down all the equilibria.

(c) 20 points. Which ones are subgame perfect?

(2) 20 points. What are the subgame perfect equilibria of the centipede game?

(3) Explain why the second player cannot force a victory in

(a) 15 points. Tic-tac-toe. Hint: assume the second player has a strategy that forces a victory. Explain how the first player can use this strategy to build a strategy that would force victory too, leading to a contradiction. This can be done in a way that is independent of most of the details of the definition of the game.

(b) 5 points. The sweet fifteen game. Hint: [https://en.wikipedia.org/wiki/Magic_square](https://en.wikipedia.org/wiki/Magic_square).

(4) *Bonus question: countability via games.* Recall that a set $S$ is countable if there exists a bijection (one-to-one correspondence) $f : S \rightarrow \mathbb{N}$ from $S$ to the natural numbers. Recall also that the interval $[0, 1]$ is not countable (Cantor, 1874). We will prove this using a game. This proof is due to Grossman and Turett (1998).

Consider the following game. Fix a subset $S \subseteq [0, 1]$, and let $a_0 = 0$ and $b_0 = 1$. The players Al and Betty take alternating turns, starting with Al. In Al’s $n^{th}$ turn he has to choose some $a_n$ which is strictly larger than $a_{n-1}$, but strictly smaller than $b_{n-1}$. At Betty’s $n^{th}$ turn she has to choose a $b_n$ that is strictly smaller than $b_{n-1}$ but strictly larger than $a_n$. Thus the sequence $(a_n)$ is strictly increasing and the sequence $(b_n)$ is strictly decreasing, and furthermore $a_n < b_m$ for every $n, m \in \mathbb{N}$.

Since $a_n$ is a bounded increasing sequence, it has a limit $a = \lim_n a_n$. Al wins the game if $a \in S$, and Betty wins the game otherwise.

(a) 1 point. Let $S$ be countable, so we can write it as $S = \{s_1, s_2, \ldots\}$. Prove that the following is a winning strategy for Betty: in her $n^{th}$ turn she chooses $b_n = s_n$ if she can (i.e., if $a_n < s_n < b_{n-1}$). Otherwise she chooses any other allowed number.

(b) 1 point. Explain why this implies that $[0, 1]$ is uncountable.