PS/EC 172, SET 1
DUE THURSDAY, OCTOBER 11^{TH}

Collaboration on homework is encouraged, but individually written solutions are required. Also, please name all collaborators and sources of information on each assignment; any such named source may be used.

(1) *Alternating ultimata*. Tanvi and Enrico are walking to lunch when they spot a \$7 note in a tree. They both quickly realize that the only way they can reach it is by having one of them climb on the shoulders of the other. It thus remains for them to agree on how they will divide the money between them once they retrieve it.

Tanvi first makes an offer to Enrico. Her offer has to be one of $\{\$0, \$1, \$2, \$3, \$4, \$5, \$6, \$7\}$, corresponding to the size of Enrico's share.

If Enrico accepts they fetch the money and split it accordingly. If Enrico rejects then he makes an offer to Tanvi. If she accepts they fetch the money and split it accordingly. Otherwise she makes an offer again, etc. At most T offers can be made before they have to go to class and the game must end. If T offers are rejected then the money is left in the tree.

(a) *20 points*. Consider the case that $T = 31415$. Construct a Nash equilibrium in which they both miss lunch and receive no money. What are their possible utilities in subgame perfect equilibria? Hint: use backward induction.

(b) *20 points*. Repeat for the case that $T = 3141592$.

(2) *Existence of equilibria*. In this question you are asked to construct extensive form games with complete information. As it turns out, these need to be infinite horizon (i.e., potentially never-ending). Note that for such games one must specify the outcomes not only after every finite terminal history, but also after every infinite one.

(a) *10 points*. Construct an extensive form game with complete information that has an action set A of size two, two players, and no Nash equilibria.

(b) *10 points*. Construct an extensive form game with complete information that has an action set A of size two, two players, a Nash equilibrium, and no subgame perfect Nash equilibria.

(3) *Deviations in infinite games*. In the lecture notes we prove (Theorem 1.7) the *one deviation principle*. Read the formal theorem statement and its proof. In this problem we will explain why we prove this claim only for finite games.

Consider the following game in which there are two players and infinitely many time periods. In the odd time periods player 1 has to decide whether to stop or continue. In the even time periods player 2 has to make the same decision. If any player decides to stop at any period then the utility is 0 for both players. If both players always continue then the utility is 1 for both.

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Consider the strategy profile s in which player 1 always stops, and player 2 always continues.

(a) *10 points*. Explain why s is not an equilibrium.

(b) *10 points*. Explain why no subgame has a profitable deviation from s that differs from s in only one move, i.e., the one deviation principle does not apply to this game.

(4) *20 points*. Consider the dollar auction game from the lecture notes, where the game ends if a bid is made that exceeds $100. Construct a subgame perfect equilibrium.

(5) *Bonus question: countability via games*. Recall that a set S is *countable* if there exists a bijection (one-to-one correspondence) $f : S \to \mathbb{N}$ from S to the natural numbers. Recall also that the interval $[0,1]$ is not countable (Cantor, 1874). We will prove this using a game. This proof is due to Grossman and Turett (1998).

Consider the following game. Fix a subset $S \subseteq [0,1]$, and let $a_0 = 0$ and $b_0 = 1$. The players Al and Betty take alternating turns, starting with Al. In Al's n^{th} turn he has to choose some a_n which is strictly larger than a_{n-1}, but strictly smaller than b_{n-1}. At Betty's n^{th} turn she has to choose a b_n that is strictly smaller than b_{n-1} but strictly larger than a_n. Thus the sequence $\{a_n\}$ is strictly increasing and the sequence $\{b_n\}$ is strictly decreasing, and furthermore $a_n < b_m$ for every $n, m \in \mathbb{N}$.

Since a_n is a bounded increasing sequence, it has a limit $a = \lim_n a_n$. Al wins the game if $a \in S$, and Betty wins the game otherwise.

(a) *1 point*. Let S be countable, so we can write it as $S = \{s_1, s_2, \ldots\}$. Prove that the following is a winning strategy for Betty: in her n^{th} turn she chooses $b_n = s_n$ if she can (i.e., if $a_n < s_n < b_{n-1}$). Otherwise she chooses any other allowed number.

(b) *1 point*. Explain why this implies that $[0,1]$ is uncountable.