PS/Ec 172, Set 5
Due Friday, May 12th at 11:59pm
Resubmission due Friday, May 26th at 11:59pm

Collaboration on homework is encouraged, but individually written solutions are required. Also, please name all collaborators and sources of information on each assignment; any such named source may be used.

(1) *The envelope paradox.* A coin is tossed until it comes out heads. Let k denotes the (random) number of tosses. There are two pieces of paper; on one is written the number 10^k, and on the other 10^{k+1}. One of the two notes is chosen at random and given to Rachel. The other is given to Max. They each look at their own note. If both want to trade then they are allowed to. After trading (or not) each is given an amount of money equal to the number written on his or her paper.

Formally, the states of the world are $\Omega = \{1, 2, 3, \ldots\} \times \{0, 1\}$, where the first coordinate is the number of tosses and the second corresponds to the random allocation of notes. There is a common prior, which, for $(k, b) \in \Omega$ is

$$\mu(k, b) = \left(\frac{1}{2}\right)^k \cdot \frac{1}{2} = 2^{-(k+1)}.$$

Rachel's signal is $t_R(k, b) = 10^{k+b}$ and Max's signal is $t_M(k, b) = 10^{k+1-b}$. Rachel's utility for not trading is t_R. Her utility for trading is t_M. Max's utility for not trading is t_M, and his utility for trading is t_R.

  (a) *40 points.* What is the common knowledge algebra Σ_C?

  (b) *40 points.* For each possible value of Rachel's signal, calculate her conditional expected gain from trading. Do the same for Max.

  (c) *20 points.* What is Rachel's expected gain from trading before she sees her signal (i.e., before she looks at her note)?

  (d) *Bonus question (1 point).* Would Rachel want to trade before she looked at her note?

(2) *Bonus question: Mind reading (with high probability).* Nachiket and Kimia play a game, in which Kimia tries to read Nachiket's mind. There is an infinite set of mailboxes $\mathbb{N} = \{1, 2, 3, \ldots\}$. Nachiket choose a finite subset $F \subset \mathbb{N}$, and places a letter in each mailbox in F. Kimia then gets to open any subset $S \subset \mathbb{N}$ of the letter boxes; note that S need not be finite, but it cannot be all of $\mathbb{N}$. She observes which ones have letters, and then has to decide to open one more box n that is not in S. Kimia wins if there is not a letter in box n, and otherwise Nachiket wins.

Formally, a pure strategy for Nachiket is a choice of a finite $F \subset \mathbb{N}$. A pure strategy for Kimia is a choice of S, plus a function $n \colon P_f(S) \to \mathbb{N} \setminus S$ from finite subsets of S to $\mathbb{N} \setminus S$. Nachiket wins if $n(S \cap F) \in F$.

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(a) *1 point.* Show that for every pure strategy of Kimia there is a pure strategy of Nachiket that ensures that he wins, and that for every pure strategy of Nachiket there is a pure strategy of Kimia that ensures that she wins.

(b) *1 point.* Show that Kimia has a *mixed* strategy (i.e., a randomly picked strategy) such that for *every* F, her probability of losing is at most $\gamma = 10^{-100}$.

(c) *1 point.* Show that Kimia has a mixed strategy such that for *every* F, her probability of losing is at most $\gamma e^{-|F|/\gamma}$.