PS/Ec 172, Homework 5
Due Thursday, November 15th

Collaboration on homework is encouraged, but individually written solutions are required. Also, please name all collaborators and sources of information on each assignment; any such named source may be used.

(1) *20 points.* Construct a belief space for two players with a common prior, a random variable P, where $Q_1 = \mathbb{E}[P|\Sigma_1]$, $Q_2 = \mathbb{E}[P|\Sigma_2]$, and in which, for some $\omega \in \Omega$, $Q_1(\omega) < q < Q_2(\omega)$ and at ω both players know that $Q_1 < q < Q_2$.

(2) *The envelope paradox.* A coin is tossed until it comes out heads. Let k denotes the (random) number of tosses. There are two pieces of paper; on one is written the number 10^k, and on the other 10^{k+1}. One of the two notes is chosen at random and given to Sophie. The other is given to Aadyot. They each look at their own note. If both want to trade then they are allowed to. After trading (or not) each is given an amount of money equal to the number written on his or her paper.

Formally, the states of the world are $\Omega = \{1, 2, 3, \ldots\} \times \{0, 1\}$, where the first coordinate is the number of tosses and the second corresponds to the random allocation of notes. There is a common prior, which, for $(k, b) \in \Omega$ is

$$\mu(k, b) = \left(\frac{1}{2}\right)^k \cdot \frac{1}{2} = 2^{-(k+1)}.$$

Sophie's type is $t_R(k, b) = 10^{k+b}$ and Aadyot's type is $t_P(k, b) = 10^{k+1-b}$. Sophie's utility for not trading is t_R. Her utility for trading is t_P. Aadyot's utility for not trading is t_P, and his utility for trading is t_R.

(a) *10 points.* What is the common knowledge algebra Σ_C?

(b) *10 points.* For each possible value of Sophie's type, calculate her conditional expected utility for trading and for not trading. Do the same for Aadyot.

(c) *10 points.* Let A be the event that both want to trade. Is it common knowledge?

(d) *10 points.* Explain the apparent conflict with the no trade theorem.

(e) *10 points.* If you had a chance to play this game, and you got to decide if a trade would take place, would you decide to trade? If so, would you look at your note before asking?

(3) *Reserve prices.* Alex and Dominic would both like to buy an item owned by Carly. Alex and Dominic's valuations are chosen independently from the uniform distribution on $[0, 1]$, and each is known only to himself.

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(a) *10 points.* What is Carly's expected revenue from a second price auction?

(b) *15 points.* Carly now introduces a *reserve price* $b_r \in [0,1]$: if the maximum bid is under b_r then the auction is canceled, no one gets the item and no one pays. Otherwise, the winner pays the maximum of b_r and the loser's bid. What is her expected revenue, as a function of b_r?

(c) *5 points.* What is the maximal expected revenue she can get by choosing b_r optimally?

(4) *Bonus: a riddle with both prisoners and hats (Gabay-O'Connor game).* There are n prisoners standing in a line. The first can observe all the rest. The second can observe all except the first, etc. Each is given either a red or a blue hat which he cannot see. Now, starting with the first prisoner, each in turn has to guess the color of his hat, a guess which the rest can hear.

(a) *1 point.* The prisoners are allowed to decide on a strategy ahead of time. Find one in which they all guess the color correctly, except maybe the first prisoner.

(b) *1 point.* Do the same, but for an infinite line of prisoners.

(c) *1 point.* For an infinite line of deaf prisoners, find a strategy in which at most finitely many of them guess incorrectly.

(d) *1 point.* For an infinite line of deaf prisoners, assume that each is assigned a hat independently and uniformly at random. Show that regardless of the strategy the prisoners agree on, each has a probability of $1/2$ of guessing his hat color correctly. Explain why this means that with probability one infinitely many prisoners will guess incorrectly. Resolve the apparent conflict with your answer from the previous question.