PS/Ec 172, Bonus Set
Due Thursday, December 6th

Collaboration on homework is encouraged, but individually written solutions are required. Also, please name all collaborators and sources of information on each assignment; any such named source may be used.

This set is optional and will count for an extra bonus point (out of 100) in the final grade.

(1) *Mind reading (with high probability).* Ali and Fatima play a game. Ali picks a finite subset $F \subset \mathbb{N}$, and Fatima picks an $n \in \mathbb{N}$. Ali wins if $n \in F$, and Fatima wins otherwise.

Before choosing her n, Fatima picks any subset $S \subseteq \mathbb{N}$. For example, S could be the even numbers. Ali reveals to Fatima the intersection $S \cap F$; we assume he does so truthfully. Fatima can now choose her number n. It can depend on Ali's answer, but it cannot be in S. She wins if $n \notin F$, and otherwise Ali wins.

Formally, a pure strategy for Ali is a choice of F. A pure strategy for Fatima is a choice of S, plus a function from subsets of S to $\mathbb{N} \setminus S$; this is the function that specifies n given $S \cap F$.

(a) Show that for every pure strategy of Fatima there is a pure strategy of Ali that ensures that he wins.

(b) Show that for every pure strategy of Ali there is a pure strategy of Fatima that ensures that she wins.

(c) Show that Fatima has a *mixed* strategy (i.e., a randomly picked strategy) such that for *every* F, her probability of winning is at least $1 - 1/2018$.

(2) *The incredible casino.* A casino has a sequence of slot machines $(M_1, M_2, \ldots)$. Each machine requires the gambler to swipe her credit card, and has a single button. After swiping the card and pressing the button, machine M_n credits the gambler 1 dollar with probability $1 - 1/n^2$, and otherwise charges her n^2 dollars.

(a) What is the gambler's expected revenue when using machine M_n?

(b) Kim gambles once at each machine, in order: M_1, M_2, M_3, etc. Explain why, with probability one, her revenue will tend to infinity.
Hint: use the Borel-Cantelli lemma. You can read about it on Wikipedia: `http://en.wikipedia.org/wiki/Borel-Cantelli_lemma`.

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