# PS/Ec 172, Set 1
## Due Friday, April 10th

Collaboration on homework is encouraged, but individually written solutions are required. Also, please name all collaborators and sources of information on each assignment; any such named source may be used.

(1) *The ultimatum game*. Consider the ultimatum game from the lecture notes.
   (a) *20 points*. How many possible strategy profiles does this game have?

   (b) *30 points*. Write down all the equilibria.

(2) Explain why the second player cannot force a victory in
   (a) *30 points*. Tic-tac-toe. Hint: assume the second player has a strategy that forces a victory. Explain how the first player can use this strategy to build a strategy that would force victory too, leading to a contradition. This can be done in a way that is independent of most of the details of the definition of the game.

   (b) *20 points*. The sweet fifteen game. Hint: https://en.wikipedia.org/wiki/Magic_square.

(3) *Bonus question: countability via games*. Recall that a set $S$ is *countable* if there exists a bijection (one-to-one correspondence) $f: S \to \mathbb{N}$ from $S$ to the natural numbers. Recall also that the interval $[0,1]$ is not countable (Cantor, 1874). We will prove this using a game. This proof is due to Grossman and Turett (1998).

   Consider the following game. Fix a subset $S \subseteq [0,1]$, and let $a_0 = 0$ and $b_0 = 1$. The players Al and Betty take alternating turns, starting with Al. In Al's $n^{\text{th}}$ turn he has to choose some $a_n$ which is strictly larger than $a_{n-1}$, but strictly smaller than $b_{n-1}$. At Betty's $n^{\text{th}}$ turn she has to choose a $b_n$ that is strictly smaller than $b_{n-1}$ but strictly larger than $a_n$. Thus the sequence $\{a_n\}$ is strictly increasing and the sequence $\{b_n\}$ is strictly decreasing, and furthermore $a_n < b_m$ for every $n, m \in \mathbb{N}$.

   Since $a_n$ is a bounded increasing sequence, it has a limit $a = \lim_n a_n$. Al wins the game if $a \in S$, and Betty wins the game otherwise.
   (a) *1 point*. Let $S$ be countable, so we can write it as $S = \{s_1, s_2, \ldots\}$. Prove that the following is a winning strategy for Betty: in her $n^{\text{th}}$ turn she chooses $b_n = s_n$ if she can (i.e., if $a_n < s_n < b_{n-1}$). Otherwise she chooses any other allowed number.

   (b) *1 point*. Explain why this implies that $[0,1]$ is uncountable.

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