Collaboration on homework is encouraged, but individually written solutions are required. Also, please name all collaborators and sources of information on each assignment; any such named source may be used.

(1) *The ultimatum game*. Consider the ultimatum game from the lecture notes.

(a) *20 points*. How many possible strategy profiles does this game have?

(b) *30 points*. Write down all the equilibria.

(2) Explain why the second player cannot force a victory in

(a) *30 points*. Tic-tac-toe. Hint: assume the second player has a strategy that forces a victory. Explain how the first player can use this strategy to build a strategy that would force victory too, leading to a contradiction. This can be done in a way that is independent of most of the details of the definition of the game.


(3) *Bonus question: countability via games*. Recall that a set $S$ is countable if there exists a bijection (one-to-one correspondence) $f:S \rightarrow \mathbb{N}$ from $S$ to the natural numbers. Recall also that the interval $[0,1]$ is not countable (Cantor, 1874). We will prove this using a game. This proof is due to Grossman and Turett (1998).

Consider the following game. Fix a subset $S \subseteq [0,1]$, and let $a_0 = 0$ and $b_0 = 1$. The players Al and Betty take alternating turns, starting with Al. In Al’s $n^{th}$ turn he has to choose some $a_n$ which is strictly larger than $a_{n-1}$, but strictly smaller than $b_{n-1}$. At Betty’s $n^{th}$ turn she has to choose a $b_n$ that is strictly smaller than $b_{n-1}$ but strictly larger than $a_n$. Thus the sequence $(a_n)$ is strictly increasing and the sequence $(b_n)$ is strictly decreasing, and furthermore $a_n < b_m$ for every $n,m \in \mathbb{N}$.

Since $a_n$ is a bounded increasing sequence, it has a limit $a = \lim_n a_n$. Al wins the game if $a \in S$, and Betty wins the game otherwise.

(a) *1 point*. Let $S$ be countable, so we can write it as $S = \{s_1, s_2, \ldots\}$. Prove that the following is a winning strategy for Betty: in her $n^{th}$ turn she chooses $b_n = s_n$ if she can (i.e., if $a_n < s_n < b_{n-1}$). Otherwise she chooses any other allowed number.

(b) *1 point*. Explain why this implies that $[0,1]$ is uncountable.

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