SS 201A, SET 1

Collaboration on homework is encouraged, but individually written solutions are required. Please name all collaborators and sources of information on each assignment. Any such named source may be used.

(1) *Alternating ultimata*. Camila and Egor are walking to lunch when they spot a \$7 note in a tree. They both quickly realize that the only way they can reach it is by having one of them climb on the shoulders of the other. It thus remains for them to agree on how they will divide the money between them once they retrieve it.

Egor first makes an offer to Camila. His offer has to be one of $\{\$0,\$1,\$2,\$3,\$4,\$5,\$6,\$7\}$, corresponding to the size of Camila's share.

If Camila accepts they fetch the money and split it accordingly. If she rejects then she makes an offer to Egor. If he accepts they fetch the money and split it accordingly. Otherwise he makes an offer again, etc. At most T offers can be made before they have to go to class and the game must end. If T offers are rejected then the money is left in the tree.

(a) Consider the case that $T = 31415$. Construct a Nash equilibrium in which they both miss lunch and receive no money. What are their possible utilities in subgame perfect equilibria? Hint: use backward induction.

(b) Repeat for the case that $T = 3141592$.

(2) Consider only extensive form games with perfect information, with at most two players and an action set of size at most two.

(a) Find a game that has no Nash equilibria.

(b) Find a game that has a Nash equilibrium but no subgame perfect equilibria.

(c) Find a game with a strategy profile s such that s is not an equilibrium, but no subgame has a profitable deviation from s that differs from s in only one move. (I.e., the one deviation principle does not apply to this game).

(3) Consider the dollar auction game as described in the lecture notes, where the game ends if a bid is made that exceeds \$100. Construct a subgame perfect equilibrium.

(4) *Intransitive dice.* A die has six sides, each labeled with a number. Consider three dice that are labeled as follows

(a) 2, 2, 4, 4, 9, 9.
(b) 1, 1, 6, 6, 8, 8.
(c) 3, 3, 5, 5, 7, 7.

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Players 1 and 2 play the following extensive form game with perfect information. First, player 1 picks one of these three dice. Then player 2 picks one of the two that are left over. The utility of a player is the probability, when the two picked dice are rolled, that their die shows the higher number.

(a) Find a subgame perfect equilibrium of this game.

(b) Who has the higher utility? Is there a subgame perfect equilibrium in which the other player has higher utility?

(c) Read this: https://en.wikipedia.org/wiki/Intransitive_dice#Warren_Buffett.

(5) Recall that a set S is *countable* if there exists a bijection (one-to-one correspondence) $f: S \to \mathbb{N}$ from S to the natural numbers. That is, if $S = \{s_1, s_2, \ldots\}$. Recall also that the interval $[0,1]$ is not countable (Cantor, 1874). We will prove this using a game. This proof is due to Grossman and Turett (1998).

Consider the following game. Fix a subset $S \subseteq [0,1]$, and let $a_0 = 0$ and $b_0 = 1$. The players Al and Betty take alternating turns, starting with Al. In Al's n^{th} turn he has to choose some a_n which is strictly larger than a_{n-1}, but strictly smaller than b_{n-1}. At Betty's n^{th} turn she has to choose a b_n that is strictly smaller than b_{n-1} but strictly larger than a_n. Thus the sequence $\{a_n\}$ is strictly increasing and the sequence $\{b_n\}$ is strictly decreasing, and furthermore $a_n < b_m$ for every $n, m \in \mathbb{N}$.

Since a_n is a bounded increasing sequence, it has a limit $a = \lim_n a_n$. Al wins the game if $a \in S$, and Betty wins the game otherwise.

(a) Let S be countable, so we can write it as $S = \{s_1, s_2, \ldots\}$. Prove that the following is a winning strategy for Betty: in her n^{th} turn she chooses $b_n = s_n$ if she can (i.e., if $a_n < s_n < b_{n-1}$). Otherwise she chooses any other allowed number.

(b) Explain why this implies that $[0,1]$ is uncountable.

(6) *Cournot competition.* The Cournot competition game is described in the lecture notes.

(a) An equilibrium is said to be *symmetric* if all players choose the same strategy. Find a symmetric pure Nash equilibrium of the Cournot competition game, as described in the lecture notes.

(b) Imagine that an organized crime boss is brought in to enforce a cartel policy that maximizes the total utility of the companies. By how much does their total utility increase?

(7) *Elimination of weakly dominated strategies.* In the following game the additional strategy A was added to matching pennies.

	H	T	A
H	1,0	0,1	2,0
T	0,1	1,0	1,0
A	1/2,0	0,1	2,2

(a) Show that this game has a pure Nash equilibrium.

(b) What are the weakly dominated strategies?

(c) Iteratively remove the weakly dominated strategies. What is the resulting game? Does it have pure Nash equilibria?

(8) *Double Brouwer.* Let $X \subset \mathbb{R}^n$ be compact and convex. Let $S, T \colon X \to X$ be continuous, and assume that they commute: $S \circ T = T \circ S$.

(a) Prove that S has a fixed point, under the assumption that $n = 1$.

(b) Show that if S has a unique fixed point then it is also a fixed point of T.

(c) *Bonus question.* Show that if T and S are affine then there is an $x \in X$ that is a fixed point of both S and T. Hint: you can use the technique used in the lecture notes to prove Brouwer for affine T.

(d) *Extra bonus question.* Find such X, S, T with the property that no $x \in X$ is a fixed point of both S and T.