SS 201A, SET 1

Collaboration on homework is encouraged, but individually written solutions are required. Please name all collaborators and sources of information on each assignment. Any such named source may be used.

(1) *Alternating ultimata*. Shunto and Danny are walking to lunch when they spot a \$7 note in a tree. They both quickly realize that the only way they can reach it is by having one of them climb on the shoulders of the other. It thus remains for them to agree on how they will divide the money between them once they retrieve it.

Shunto first makes an offer to Danny. His offer has to be one of $\{\$0, \$1, \$2, \$3, \$4, \$5, \$6, \$7\}$, corresponding to the size of Danny's share.

If Danny accepts they fetch the money and split it accordingly. If Danny rejects then he makes an offer to Shunto. If he accepts they fetch the money and split it accordingly. Otherwise Shunto makes an offer again, etc. At most T offers can be made before they have to go to class and the game must end. If T offers are rejected then the money is left in the tree.

(a) Consider the case that $T = 31415$. Construct a Nash equilibrium in which they both miss lunch and receive no money. What are their possible utilities in subgame perfect equilibria? Hint: use backward induction.

(b) Repeat for the case that $T = 3141592$.

(2) Consider only extensive form games with perfect information, with two players and an action set of size two.

(a) Find a game that has no Nash equilibria.

(b) Find a game that has a Nash equilibrium but no subgame perfect equilibria.

(c) Find a game with a strategy profile s such that s is not an equilibrium, but no subgame has a profitable deviation from s that differs from s in only one move. (I.e., the one deviation principle does not apply to this game).

(3) Consider the dollar auction game, where the game ends if a bid is made that exceeds \$100. Construct a subgame perfect equilibrium. Is there one for the case of no limit on the bids?

(4) Recall that a set S is *countable* if there exists a bijection (one-to-one correspondence) $f : S \to \mathbb{N}$ from S to the natural numbers. Recall also that the interval $[0,1]$ is not countable (Cantor, 1874). We will prove this using a game. This proof is due to Grossman and Turett (1998).

Consider the following game. Fix a subset $S \subseteq [0,1]$, and let $a_0 = 0$ and $b_0 = 1$. The players Al and Betty take alternating turns, starting with Al. In

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Al's n^{th} turn he has to choose some a_n which is strictly larger than a_{n-1}, but strictly smaller than b_{n-1}. At Betty's n^{th} turn she has to choose a b_n that is strictly smaller than b_{n-1} but strictly larger than a_n. Thus the sequence $\{a_n\}$ is strictly increasing and the sequence $\{b_n\}$ is strictly decreasing, and furthermore $a_n < b_m$ for every $n, m \in \mathbb{N}$.

Since a_n is a bounded increasing sequence, it has a limit $a = \lim_n a_n$. Al wins the game if $a \in S$, and Betty wins the game otherwise.

(a) Let S be countable, so we can write it as $S = \{s_1, s_2, \ldots\}$. Prove that the following is a winning strategy for Betty: in her n^{th} turn she chooses $b_n = s_n$ if she can (i.e., if $a_n < s_n < b_{n-1}$). Otherwise she chooses any other allowed number.

(b) Explain why this implies that $[0, 1]$ is uncountable.