SS 205B, SET 1

DUE MONDAY, JANUARY 18TH

Collaboration on homework is encouraged, but individually written solutions are required. Also, please name all collaborators and sources of information on each assignment; any such named source may be used.

Note: some of the claims that you are asked to prove are false. For these claims please provide a counterexample.

(1) Let X be a subset of $\mathbb{R}^n$. Recall that $D \subseteq X$ is dense in X if for every $x \in X$ and every $\varepsilon > 0$ there is a $d \in D$ such that $\|x - d\| < \varepsilon$. Prove that every subset of $\mathbb{R}^n$ has a countable dense subset.

(2) Consider a consumer with a closed consumption set $X \subseteq \mathbb{R}^L$ and a closed preference $\succ$ on X. Recall that given $p \in \mathbb{R}^L$ and $w \in \mathbb{R}$, we denote

$$X^*(p,w) = \{x^* \in X : p \cdot x^* \leq w \text{ and } p \cdot x \leq w \text{ implies } x^* \succeq x\}.$$

Recall also that $\succ$ is said to be locally non-satiated (LNS) if for every $\varepsilon >$ and $x \in X$ there is a y such that $\|x - y\| \leq \varepsilon$ and $y \succ x$.

Finally, recall that $\succ$ is said to be convex if $x' \succeq x$ and $x'' \succeq x$ implies $z \succeq x$ for all $\alpha \in [0,1]$ and $z = \alpha x' + (1-\alpha)x'' \in X$.

(a) Show that if $\succ$ is LNS, and if X is connected, then X cannot be compact. Hint: use the theorem stated in class which guarantees that $\succ$, as a closed preference on the closed connected set X, is represented by a continuous utility function $u : X \to \mathbb{R}$.

(b) Show that if $\succ$ is LNS and if $x^* \in X^*(p,w)$ then $p \cdot x^* = w$, and that, more generally, if $x \succeq x^*$ then $p \cdot x \geq w$.

(c) Show that if $\succ$ is convex and X is convex then $X^*(p,w)$ is convex (whenever it is non-empty).

(3) Let $(X_i)_i$, $(\succ_i)_i$ and $(e_i)_i$ be the consumption sets, preferences and endowments of an exchange economy. Suppose that each X_i is closed and that each $\succ_i$ is closed and LNS.

Recall that $(x_i)_i$ and p form a competitive equilibrium if $x_i \in X_i^*(p, p \cdot e_i)$ for all i, and if $\sum_i x_i = \sum_i e_i$.

Recall also that $(x_i)_i$ is Pareto optimal if $x_i \in X_i$ for all i, and if for every $(x_i')_i$ such that $\sum_i x_i' = \sum_i e_i$ and $x_i' \succeq_i x_i$ for all i it holds that $x_i \succeq x_i'$ for all i.

(a) Prove that if $(x_i)_i$ and p form an equilibrium then $(x_i)_i$ is Pareto optimal.

(b) Prove that if $(x_i)_i$ is Pareto optimal then there exists a p such that $(x_i)_i$ and p form an equilibrium.

(4) Recall that the support function of a compact convex $Y \subset \mathbb{R}^n$ is given by $v_Y(p) = \max_{y \in Y} p \cdot y$.

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(a) Prove that $Y' \subseteq Y$ iff $v_{Y'}(p) \leq V_Y(p)$ for all p.

(b) For $\lambda > 0$ and $Y \subseteq \mathbb{R}^n$ denote $\lambda Y = \{\lambda y, y \in Y\}$. Prove that $v_{\lambda Y}(p) = \lambda v(p)$.