Foundations of Economics
Lecture Notes

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Contents

1 Preferences 4

2 Prices and consumer choice 6

3 Production 7

4 Private ownership economies and Walrasian equilibria 8

5 Walrasian equilibria with transfers 10

6 Excess demand 13

7 Compactifying the economy 15

8 Kakutani's Theorem and Debreu's Theorem 16

9 Existence of equilibria 17

10 Approximate equilibria 18

11 Scitovsky contours 21

12 The core 22

13 The core via approximate equilibria 25

14 Partial equilibrium: consumers 28

15 Partial equilibrium: production 30

16 Partial equilibrium conditions 31

17 Uncertainty 33

18 Pari-mutuel gambling 34

19 Radner equilibria 39

20 Asset pricing 42

Disclaimer

This a not a textbook. These are lecture notes.

1 Preferences

Let X be a metric space. We say that X is connected if it is not the disjoint union of two closed sets. A closed-contour preference on X is a reflexive, transitive and complete binary relation $\preceq$ such that for each $x \in X$ the *upper contour set* $\{x' \in X : x \preceq x'\}$ and *lower contour set* $\{x' \in X : x' \preceq x\}$ are closed subsets of X. We say that $\preceq$ is a closed preference if it is a closed subset of $X \times X$; that is if $\{(x,x') : x \preceq x'\}$ is a closed subset of $X \times X$.

We say that a subset $Q \subseteq X$ is *dense* if for each $x \in X$ there is a sequence $(q_n)_n$ in Q with $\lim_n q_n = x$. Equivalently, X is equal to the closure of Q. Recall that X is *separable* if it admits a countable dense.

Theorem 1.1 (Debreu)**.** *For every closed-contour preference $\preceq$ on a separable, connected metric space X there is a continuous function $u : X \to \mathbb{R}$ such that $x \preceq x'$ iff $u(x) \le u(x')$.*

Corollary 1.2. *Let X be a connected, separable metric space, and let $\preceq$ be a preference on X. Then the following are equivalent:*

1. *$\preceq$ is closed-contour.*
2. *$\preceq$ is closed.*

Proof. Suppose $\preceq$ is closed-contour. Then by Theorem 1.1 there is a continuous utility function $u : X \to \mathbb{R}$ that represented $\preceq$. Suppose $x_n \preceq x'_n$ for $n \in \{1,2,\ldots\}$. Then $u(x_n) \le u(x'_n)$. If the two sequences converge to x and x' respectively, then $u(x) \le u(x')$, and thus $\{(x,x') : x \preceq x'\}$ is closed.

Conversely, assume $\preceq$ is closed. Then $\{x' : x' \preceq x\} = \{(w,w') : w \preceq w'\} \cap X \times \{x\}$ is the intersection of two closed sets, and is therefore closed. □

We will henceforth assume that all preferences are closed (equivalently, contour-closed).

We say that a preference $\preceq$ on a metric space X is *locally non-satiated* (LNS) if for every $x \in X$ and $\varepsilon > 0$ there is a x' such that $d(x,x') < \varepsilon$ and $x' \succ x$.

We say that $\preceq$ is *convex* if $\{x' : x' \succeq x\}$ is convex for all $x \in X$. We say that it is *strictly convex* if $\lambda w + (1-\lambda)w' \succ x$ whenever $w \succeq x$ and $w' \succeq x$.

Claim 1.3. *If $\preceq$ is convex then $\{x' : x' \succ x\}$ is convex for all $x \in X$.*

Suppose X is convex. We will say that $\preceq$ is *convex** if

$$x' \succ x \quad \text{implies} \quad \lambda x' + (1-\lambda)x \succ x \;\; \text{for all} \lambda \in (0,1). \tag{1.1}$$

Claim 1.4. *If X is convex and $\preceq$ is convex* then $\preceq$ is convex.*

Proof. Let $\preceq$ be represented by $u : X \to \mathbb{R}$. We claim that (1.1) implies that u is *quasiconcave*, i.e.,

$$u(\lambda x + (1-\lambda)x') \ge \min\{u(x), u(x')\}. \tag{1.2}$$

Suppose not, so that the above inequality is violated for some x, x' and λ. Denote $v(\alpha) = u(\alpha x + (1-\alpha)x')$, and assume without loss of generality that $v(0) \le v(1)$. By assumption we have that $v(\lambda) < v(0)$. It follows from the continuity of v that there is some $\beta \in (0, \lambda)$ such that $v(\lambda) < v(\beta) < v(0) \le v(1)$. But since $\lambda \in (\beta, 1)$, it follows from (1.1) that $v(\lambda) > v(\beta)$, and we have reached a contradiction.

To complete the proof we claim that the quasiconcavity of u implies (in fact, is equivalent to) the convexity of $\preceq$. Indeed, suppose that $x' \succeq x$ and $x'' \succeq x$, and assume without loss of generality that $x' \succeq x''$. By the quasiconcavity condition (1.2) $\hat{x} = \lambda x' + (1-\lambda)x''$ satisfies $\hat{x} \succeq x''$, and so $\hat{x} \succeq x$ (the proof of the other direction is identical). □

Claim 1.5. *Suppose X is convex, $\preceq$ is convex*, and for every $x \in X$ there exists $x' \in X$ with $x' \succ x$. Then $\preceq$ is LNS.*

Proof. For each $x \in X$, let $x' \succ x$. The sequence $x^n = \frac{n-1}{n}x + \frac{1}{n}x'$ satisfies $x^n \succ x$ and intersects every ball around x. □

2 Prices and consumer choice

We consider a setting with L commodities $\{1,\ldots,L\}$. A *consumption space* X is a subset of $\mathbb{R}^L$. Let $\preceq$ be a preference on X.

A price vector p is an element of $\mathbb{R}^L$. Given p and *wealth* $w \in \mathbb{R}$, the choice set $X^*(p,w)$ is given by

$$X^*(p,w) = \{x \in X : p \cdot x' \leq w \text{ implies } x' \preceq x\}. \tag{2.1}$$

Equivalently, $X^*(p,w) = \{x \in X : x' \succ x \text{ implies } p \cdot x' > w\}$.

Lemma 2.1. *Suppose $\preceq$ is LNS, and $x^* \in X^*(p,w)$. Then $x \succeq x^*$ implies $p \cdot x^* \geq w$.*

Proof. Suppose towards a contradiction that $x \succeq x^*$ and $p \cdot x < w$. Then there is some ε small enough so that $p \cdot x' < w$ for all x' such that $\|x - x'\|\varepsilon$. Since $\preceq$ is LNS, there is some such x' such that $x' \succ x$. Hence $x' \succ x^*$, and so it is impossible that $x^* \in X^*(p,w)$. □

Claim 2.2. *Suppose X is convex. Fix a price vector p, $w \in \mathbb{R}$ and $x \in X$ such that $x' \succ x$ implies $p \cdot x' \geq w$, and such that there is some $\hat{x} \in X$ with $p \cdot \hat{x} < w$. Then $x \in X^*(p,w)$.*

Proof. Let $x' \succ x$. We need to show that $p \cdot x' > w$. By assumption $p \cdot x' \geq w$, and so it remains to show that $p \cdot x' \neq w$. Assume $p \cdot x' = w$. Since $\prec$ is closed and X is convex, for all λ small enough it holds that $x'' = (1-\alpha)x + \alpha\hat{x} \succ x$. But for λ small enough it also holds that $p \cdot x'' < w$, in contradiction to the claim hypothesis. □

3 Production

In a setting with L commodities, a *production set* Y is a subset of $\mathbb{R}^L$. Given a price vector p, the optimal production set is

$$Y^*(p) = \{y \in Y : p \cdot y' \le p \cdot y \text{ for all } y' \in Y\}. \tag{3.1}$$

The *indirect utility* is given by

$$v_Y(p) = \sup_{y \in Y} p \cdot y. \tag{3.2}$$

This is also known as the *support function* of Y. Note that $v_Y(p) = p \cdot y$ for all $y \in Y^*(p)$.

For $\lambda \ge 0$, we denote $\lambda Y = \{\lambda y : y \in Y\}$. The *Minkowski sum* of two non-empty subsets $Y_1, Y_2 \subseteq \mathbb{R}^L$ is given by $Y_1 + Y_2 = \{y_1 + y_2 : y_1 \in Y_1, y_2 \in Y_2\}$. It is easy to see that

1. $v_{\lambda Y} = \lambda v_Y$.

2. $v_{Y_1} + v_{Y_2} = v_{Y_1+Y_2}$.

The Minkowski sum of two closed sets is in general not closed. However, if Y_1 is open, then $Y_1 + Y_2$ is open for every non-empty Y_2, since it is equal to the union of open sets $\bigcup_{y_2 \in Y_2} Y_1 + \{y_2\}$.

We denote *Minkowski subtraction* by

$$Y_1 - Y_2 = \{y_1 - y_2 : y_1 \in Y_1, y_2 \in Y_2\}.$$

Claim 3.1. *Denote $Y = Y_1 + \cdots + Y_L$. The following are equivalent:*

1. $\sum_j y_j \in Y^*(p)$.

2. $y_j \in Y_j^*(p)$ *for* $j = 1, \ldots, J$.

Fix some price vector p, and let $y \in Y^*(p)$. How does y change when we change p? That is, if $y' \in Y^*(p')$, what can we say about the relation between y and y', given p and p'?

Claim 3.2. *Let $y \in Y^*(p)$ and $y' \in Y^*(p')$. Denote $\Delta y = y' - y$ and $\Delta p = p' - p$. Then*

$$\delta p \cdot \Delta y \ge 0.$$

Proof. Since $p \cdot y' \le p \cdot y$ we have that

$$p \cdot \Delta y \le 0$$

Likewise, $p' \cdot y' \ge p' \cdot y$, and so

$$p' \cdot \Delta y \ge 0.$$

Subtracting the first from the second yields the desired result. □

4 Private ownership economies and Walrasian equilibria

A private ownership economy consists of the following elements:

1. L commodities.
2. I consumers, each with a consumption set $X_i \subset \mathbb{R}^L$, a preference $\preceq_i$ on X_i, and an endowment $e_i \in \mathbb{R}^L$.
3. J firms, each with a production set Y_j.
4. Each consumer i holds a stake θ_{ij} in firm j. We assume $\sum_i \theta_{ij} = 1$ for all j.

An *allocation* is a pair $((x_i)_i, (y_j)_j)$ such that $x_i \in X_i$, $y_j \in Y_j$ and $\sum_i x_i = \sum_i e_i + \sum_j y_j$. The set of all allocations is denoted by A.

A *Walrasian equilibrium* consists of a price vector p together with consumption vectors $(x_i)_i$ and production vectors $(y_j)_j$ such that

1. For all i, $x_i \in X_i^*(p, w_i)$, where $w_i = p \cdot e_i + \sum_j \theta_{ij} p \cdot y_j$.
2. For all j, $y_j \in Y_j^*(p)$.
3. $((x_i)_i, (y_j)_j)$ is an allocation.

We say that $(x_i)_i$ is *Pareto optimal* if there exists no allocation $((x'_i)_i, (y'_i)_i)$ such that $x'_i \succeq x_i$ for all i, and $x'_\ell \succ x_\ell$ for some ℓ.

Theorem 4.1 (First Welfare Theorem)**.** *Suppose each $\preceq_i$ is LNS, and that p, $(x_i)_i$ and $(y_j)_j$ form an equilibrium. Then x_i is Pareto optimal.*

Proof. Suppose x_i is not Pareto optimal. Then there exists an allocation $((x'_i)_i, (y'_j)_j)$ such that $x'_i \succeq x_i$ for all i, and $x'_\ell \succ x_\ell$ for some ℓ. Then, by the first equilibrium condition, $p \cdot x'_\ell > w_i$, and by Lemma 2.1, $p \cdot x'_i \geq w_i$ for all i. Thus

$$\sum_i p \cdot x'_i > \sum_i w_i.$$

Substituting the definition of w_i yields

$$\sum_i p \cdot x'_i > \sum_i \left(p \cdot e_i + \sum_j \theta_{ij} p \cdot y_j \right).$$

Since $y_j \in Y_j^*(p)$, we know that $p \cdot y_j \geq p \cdot y'_j$ by the second equilibrium condition, and so

$$\sum_i p \cdot x'_i > \sum_i \left(p \cdot e_i + \sum_j \theta_{ij} p \cdot y'_j \right).$$

Taking p out of the parentheses yields

$$p \cdot \left(\sum_i x'_i \right) > p \cdot \left(\sum_i \left(e_i + \sum_j \theta_{ij} y'_j \right) \right),$$

and then changing the order of summation we arrive at

$$p \cdot \left(\sum_i x'_i \right) > p \cdot \left(\sum_i e_i + \sum_j y'_j \sum_i \theta_{ij} \right).$$

Recall that $\sum_i \theta_{ij} = 1$, and so

$$p \cdot \left(\sum_i x'_i \right) > p \cdot \left(\sum_i e_i + \sum_j y'_j \right),$$

which contradicts the assumption that $((x'_i)_i, (y'_j)_j)$ is an allocation. □

5 Walrasian equilibria with transfers

A *Walrasian equilibrium with transfers* consists of a price vector p and a transfers vector t, together with consumption vectors $(x_i)_i$ and production vectors $(y_j)_j$ such that

1. For all i, $x_i \in X_i^*(p, w_i)$, where $w_i = t_i + p \cdot e_i + \sum_j \theta_{ij} p \cdot y_j)$.
2. For all j, $y_j \in Y_j^*(p)$.
3. $((x_i)_i, (y_j)_j)$ is an allocation.
4. $\sum_i t_i = 0$.

A *Walrasian quasi-equilibrium with transfers* consists of a price vector p and a transfers vector t, together with consumption vectors $(x_i)_i$ and production vectors $(y_j)_j$ such that

1. For all i, $x_i' \succ_i x_i$ implies $p \cdot x_i' \geq w_i$, where $w_i = t_i + p \cdot e_i + \sum_j \theta_{ij} p \cdot y_j$.
2. For all j, $y_j \in Y_j^*(p)$.
3. $((x_i)_i, (y_j)_j)$ is an allocation.
4. $\sum_i t_i = 0$.

Theorem 5.1 (Second Welfare Theorem)**.** *Suppose that each $\preceq_i$ is LNS and convex, each X_i and Y_j is convex, $((x_i)_i, (y_j)_j)$ is an allocation, and that $(x_i)_i$ is Pareto optimal. Then there exists a price vector p and a transfer vector t such that p, t, $(x_i)_i$ and $(y_j)_j$ form a quasi-equilibrium with transfers, and an equilibrium under the additional assumption that each x_i is in the interior of X_i.*

Proof. Denote $x = \sum_i x_i$, $y = \sum_j y_j$ and $e = \sum_i e_i$. Let $X_i^+ = \{x_i' : x_i' \succ x_i\}$. Note that this is a non-empty set since $\preceq_i$ is LNS, an open set since $\preceq$ is closed, and a convex set, by Claim 1.3. Denote

$$X^+ = \sum_i X_i^+.$$

As a Minkowski sum of convex sets, X^+ is also convex. Likewise, denote

$$Y = \sum_j Y_j$$

and

$$G^+ = X^+ - Y,$$

which are again convex, as a Minkowski sums of convex sets. Note also that G^+ is open, since X^+ is open, and the sum of an open set with any set is again open.

We claim that $e \not\in G^+$, since otherwise there will be an allocation $((x_i^+)_i, (y_j^+)_j)$ such that $x_i^+ \succ_i x_i$ for all i, in contradiction to the assumption that $(x_i)_i$ is Pareto optimal. Hence, by the Separating Hyperplane Theorem, there is some (nonzero) price vector $p \in \mathbb{R}^L$ and a $w \in \mathbb{R}$ such that $p \cdot e = w$ and $p \cdot z > w$ for all $z \in G^+$. Fix some such p.

By LNS, e is in $\overline{G^+}$, the closure of G^+. This is because LNS implies that for each i there exists a sequence $(x_{i,n}^+)_n$ such that $x_{i,n}^+ \succ_i x_i$ and $x_{i,n}^+ \to_n x_i$. Thus e is a minimizer of $p \cdot z$ on $\overline{G^+}$. We claim that it follows that $y \in Y^*(p)$. Assume to the contrary that $p \cdot y' > p \cdot y$, and let $e' = x - y'$. Then $e' \in \overline{G^+}$, and

$$p \cdot e' = p \cdot x - p \cdot y' < p \cdot x - p \cdot y = p \cdot e,$$

in contradiction to our observation that e minimizes $p \cdot z$ on $\overline{G^+}$. Thus, by Claim 3.1, $y_j \in Y_j^*(p)$ for all j.

Let

$$t_i = p \cdot x_i - (p \cdot e_i + \sum_j \theta_{ij} p \cdot y_j).$$

Then

$$\sum t_i = p \cdot x - (p \cdot e + \sum_i \sum_j \theta_{ij} p \cdot y_j) = p \cdot x - (p \cdot e + \sum_j p \cdot y_j) = 0,$$

where the first equality is s substitution of the definitions of x and e, the second a consequence of $\sum_i \theta_{ij} = 1$, and the third follows because $((x_i)_i, (y_j)_j)$ is an allocation.

By our definition of t_i, $p \cdot x_i = w_i$. To show that we have constructed a quasi-equilibrium we show that if $x_i' \succ_i x_i$ then $p \cdot x_i' \geq w_i$. Suppose towards a contradiction that there is an $x_\ell' \in X_\ell$ such that $x_\ell' \succ_\ell x_\ell$ and $p \cdot x_\ell' < w_\ell - \varepsilon$ for some $\varepsilon > 0$. By LNS, we can find for each $i \neq \ell$ an $x_i' \in X_i$ such that $x_i' \succ_i x_i$ and $p \cdot x_i' < w_i + \varepsilon/I$. It follows that

$$p \cdot \sum_i x_i' < p \cdot \sum_i x_i.$$

Denote $x' = \sum_i x_i'$. Then $x' - y \in G^+$, and yet $p \cdot (x' - y) < p \cdot (x - y) = p \cdot e$, and so we have reached a contradiction. Thus we have constructed a quasi-equilibrium.

Finally, suppose that each x_i is an interior point. Then for each x_i there is an $\hat{x}_i$ such that $p \cdot \hat{x}_i < w_i$. It thus follows from Claim 2.2 that $x_i \in X^*(p, w_i)$, and so we have an equilibrium. □

Proof. Denote $x = \sum_i x_i$, $y = \sum_j y_j$ and $e = \sum_i e_i$. Let $X_i^+ = \{x_i' : x_i' \succ x_i\}$. Note that this is a non-empty set since $\preceq_i$ is LNS, an open set since $\preceq$ is closed, and a convex set, by Claim 1.3. Denote

$$X^+ = \sum_i X_i^+.$$

As a Minkowski sum of convex sets, X^+ is also convex. Furthermore, as a sum of open sets it is open.

Likewise, denote

$$\hat{Y} = \{e\} + \sum_j Y_j,$$

and note that $\hat{Y}$ is convex.

We claim that X^+ and $\hat{Y}$ are disjoint. Suppose not, so that there is some $(x_i^+)_i$ and $(y_j)_j$ such that $\sum_i x_i^+ = e + \sum_j y_j$. But then this is an allocation with consumptions that strictly dominates $(x_i)_i$ for each i, which contradicts the assumption that $(x_i)_i$ is Pareto optimal.

It follows that by the Separating Hyperplane Theorem there is some (nonzero) price vector $p \in \mathbb{R}^L$ and a $w \in \mathbb{R}$ such that $p \cdot (e + y) \leq w$ for all $e + y \in \hat{Y}$ and such that $p \cdot x^+ > w$ for all $x^+ \in X^+$. Fix some such p. Note that $x \in \hat{Y}$, since $((x_i)_i, (y_j)_j)$ is an allocation. Hence $p \cdot x \leq w$. On the other hand, by LNS, x is in the closure of X^+ and so $p \cdot x \geq w$. We thus have that $w = p \cdot x$. Since $((x_i)_i, (y_j)_j)$ is an allocation, $w = p \cdot (e + y)$.

We claim first that $y_j \in Y_j^*(p)$ for all j. By Claim 3.1, this is equivalent to $p \cdot y' \leq p \cdot y$ for any $y' \in \sum_j Y_j$. This in turn is equivalent to $p \cdot (e + y') \leq p \cdot (e + y) = w$, which we have already shown above.

Let

$$t_i = p \cdot x_i - (p \cdot e_i + \sum_j \theta_{ij} p \cdot y_j).$$

Then

$$\sum t_i = p \cdot x - (p \cdot e + \sum_i \sum_j \theta_{ij} p \cdot y_j) = p \cdot x - (p \cdot e + \sum_j p \cdot y_j) = 0,$$

where the first equality is a substitution of the definitions of x and e, the second a consequence of $\sum_i \theta_{ij} = 1$, and the third follows because $((x_i)_i, (y_j)_j)$ is an allocation.

By our definition of t_i, $p \cdot x_i = w_i$. To show that we have constructed a quasi-equilibrium we show that if $x_i' \succ_i x_i$ then $p \cdot x_i' \geq w_i$. Suppose towards a contradiction that there is an $x_\ell^+ \in X_\ell$ such that $x_\ell^+ \succ_\ell x_\ell$ and $p \cdot x_\ell^+ < w_\ell - \varepsilon$ for some $\varepsilon > 0$. By LNS, we can find for each $i \neq \ell$ an $x_i^+ \in X_i$ such that $x_i^+ \succ_i x_i$ and $p \cdot x_i^+ < w_i + \varepsilon / I$. It follows that

$$p \cdot \sum_i x_i^+ < p \cdot \sum_i x_i.$$

Denote $x^+ = \sum_i x_i^+$. Then $x^+ \in X^+$, and yet $p \cdot x^+ < p \cdot x = p \cdot e$, and so we have reached a contradiction. Thus we have constructed a quasi-equilibrium.

Finally, suppose that each x_i is an interior point. Then for each x_i there is an $\hat{x}_i$ such that $p \cdot \hat{x}_i < w_i$. It thus follows from Claim 2.2 that $x_i \in X^*(p, w_i)$, and so we have an equilibrium.

□

6 Excess demand

Consider a private ownership economy. We denote $X = \sum_i X_i$, $Y = \sum_j Y_j$ and $\sum_i e_i = e$. Let p be a price vector. Recall that consumer i with wealth w_i will consume a bundle in $X_i^*(p,w)$. Recall also that for each price vector p, $Y_j^*(p)$ is the set of all $y_j^* \in Y_j$ that maximize $p \cdot y_j$. Note that $p \cdot y_j^* = p \cdot \bar{y}_j^*$ for all $y_j^*, \bar{y}_j^* \in Y_j^*(p)$, and so $p \cdot Y_j^*(p)$ is well defined, as long as $Y_j^*(p)$ is non-empty. The wealth of consumer i depends on p and is given by

$$w_i(p) = p \cdot e_i + \sum_j \theta_{ij} p \cdot Y_j^*(p).$$

The set of optimal consumption bundles can thus be written as depending on p alone:

$$X_i^*(p) = X_i^*(p, w_i(p)).$$

Let

$$X^*(p) = \sum_i X_i^*(p) \quad \text{and} \quad Y^*(p) = \sum_j Y_j^*(p).$$

We define the *excess demand* at price p by

$$Z^*(p) = X^*(p) - Y^*(p) - e.$$

The excess demand at rice p is a set of consumption bundles. Each corresponds to a possible total amount that is consumed at that price, in excess of the total available - i.e., produced plus endowed.

We note that for some p either $X_i^*(p)$ or $Y_j^*(p)$ might be empty. In that case we also define $Z^*(p)$ to be empty. We denote by $\mathscr{P}$ the set of prices for which $Z^*(p)$ is non-empty.

It is easy to see that if t is a positive number, and if $p \in \mathscr{P}$, then $Z^*(tp) = Z^*(t)$. If all prices are multiplied by a positive constant then neither producers nor consumers change their behavior: $X_i^*(tp) = X_i^*(p)$ and $Y_j^*(tp) = Y_j^*(p)$. Thus also $Z^*(tp) = Z^*(p)$. A function or correspondence with this property is called *homogeneous of degree zero*.

Claim 6.1 (*Walras's Law*). *$p \cdot z \leq 0$ for all $z \in Z^*(p)$. If we furthermore assume that each $\preceq_i$ is LNS, then this holds with equality.*

Proof. Let $z = \sum_i x_i^* - \sum_j y_j^* - \sum_i e_i$, with $x_i^* \in X_i^*(p)$ and $y_j^* \in Y_j^*(p)$. Then by the definition of X_i^* we have that

$$p \cdot x_i^* \leq p \cdot e_i + \sum_j \theta_{ij} p \cdot y_j^*(p).$$

Summing over i yields

$$p \cdot \sum_i x_i^* \leq p \cdot \sum_i e_i + \sum_i \sum_j \theta_{ij} p \cdot y_j^*(p).$$

Since $\sum_i \theta_{ij} = 1$ for all j, this implies

$$p \cdot \sum_i x_i^* \leq p \cdot \sum_i e_i + p \cdot \sum_j y_j^*(p),$$

or

$$p \cdot z \leq 0.$$

Finally, by Lemma 2.1, all the above inequalities hold with equality when each $\preceq_i$ is LNS. □

We say that there is *free disposal* if Y contains $\mathbb{R}_-^L$.

Claim 6.2. *If there is free disposal then* $\mathscr{P} \subseteq \mathbb{R}_+^L$.

Proof. Choose any p with $p_\ell < 0$ for some commodity ℓ, and let $y^n \in Y$ equal $-n$ in coordinate ℓ, and vanish in the remaining coordinates. Then $p \cdot y^n = |p_\ell| \cdot n$, and so there is no $y^* \in Y$ that maximizes $p \cdot y$ in Y. Thus $Y^*(p)$ is empty, and so $Z^*(p)$ is empty. □

Clearly, an equilibrium exists iff there is a p such that $0 \in Z^*(p)$. The next claim shows that under some conditions, it suffices that $Z^*(p)$ has an element in $\mathbb{R}_-^L$.

Claim 6.3. *If there is free disposal (i.e., $Y \supseteq \mathbb{R}_-^L$), if Y is convex, and if each $\preceq_i$ is LNS, then an equilibrium exists iff there is a p such that $Z^*(p)$ intersects $\mathbb{R}_-^L$.*

Proof. If there is an equilibrium p, $(x_i)_i$, $(y_j)_j$ then $z = \sum_i x_i - \sum_j y_j - \sum_i e_i = 0$ is in $Z^*(p)$, by definition. Hence $Z^*(p)$ intersects $\mathbb{R}_-^L$.

Suppose $z \in Z^*(p)$ and $z \in \mathbb{R}_-^L$, and write $z = x - y - e$, with $x = \sum_i x_i$, $x_i \in X_i^*(p, w_i(p))$, and $y = \sum_j y_j$, $y_j \in Y_j^*(p)$. Let $y' = y + z$, and note that $y' \in Y$, since $Y \supseteq -\mathbb{R}^L$ and Y is convex. By Claim 6.1, $p \cdot z = 0$. Hence

$$p \cdot y' = p \cdot y + p \cdot z = p \cdot y,$$

and so $y' \in Y^*(p)$. Let $z' = x - y' - e$, and note that $z' \in Z^*(p)$. Finally,

$$z' = x - y' - e = x - (y + z) - e = x - (y + x - y - e) - e = 0.$$

Hence p, $(x_i)_i$ and $(y'_j)_j$ form an equilibrium. □

7 Compactifying the economy

Recall that an allocation $((x_i)_i, (y_j)_j)$ is an element of $(\prod_i X_i) \times (\prod_j Y_k)$ such that $\sum_i x_i = \sum_i e_i + \sum_j y_j$, and that the set of all allocations is denoted by A.

Claim 7.1. *Suppose that $Y \cap \mathbb{R}^L_+ = \{0\}$, $Y \cap (-Y) = \{0\}$, Y is convex and $X_i \subseteq \mathbb{R}^L_+$. Then A is bounded.*

Proof. Note that if Y'_j is a superset of Y_j and X'_i is a superset of X_i, then the corresponding A' contains A. We can thus always enlarge X_i and Y_j without loss of generality. We will accordingly assume that $X_i = \mathbb{R}^L_+$ and that each Y_j is a closed cone, by adding to it any elements of the form λy_j, for $\lambda > 0$ and $y_j \in Y_j$, and taking the closure.

Suppose towards a contradiction that for every n we can find $a^n = ((x^n_i)_i, (y^n_j)_j)$ such that $a^n \in A$ and

$$\|a^n\|^2 = \sum_i \|x^n_i\|^2 + \sum_j \|y^n_j\|^2$$

tends to infinity. Since each X_i and Y_j are cones,

$$\hat{x}^n_i = x^n_i / \|a^n\| \quad \text{and} \quad \hat{y}^n_i = y^n_j / \|a^n\|$$

are both also in X_i and Y_j respectively. The sequence $\hat{a}^n = ((\hat{x}^n_i)_i, (\hat{y}^n_j)_j)$ of unit vectors has a converging subsequence, and so we assume without loss of generality that it converges to some unit vector $\hat{a} = ((\hat{x}_i)_i, (\hat{y}_j)_j)$. Since X_i and Y_j are closed we have that $\hat{x}_i \in X_i$ and $\hat{y}_j \in Y_j$. Since $x^n - y^n = e$, $\hat{x}^n - \hat{y}^n = e/\|a\|$, and so $\hat{x} - \hat{y} = 0$.

Since $Y \cap X = \{0\}$ we get that $\hat{x} = \hat{y} = 0$. It follow immediately that $\hat{x}_i = 0$ for all i. We claim that likewise $\hat{y}_j = 0$ for all j. Otherwise, suppose $y_\ell \neq 0$. Then $\sum_{j \neq \ell} y_j = -y_\ell$. Since each Y_j is a cone, y_ℓ and $-y_\ell$ are both in Y, in contradiction to the assumption that $Y \cap (-Y) = \{0\}$. We have thus shown that $\hat{a} = 0$, in contradiction to the fact that $\hat{a}$ is a unit vector. □

Proposition 7.2. *Suppose that each Y_j and X_i is convex, each $\preceq_i$ is convex*, and that A is bounded. Let K be a compact convex set whose interior contains A. Define a "hat" private ownership economy by $\hat{X}_i = K \cap X_i$ and $\hat{Y}_i = K \cap Y_i$, with all other elements remaining the same. Then every equilibrium of the hat economy is an equilibrium of the original economy.*

Proof. Suppose p, $(\hat{x}_i)_i$ and $(\hat{y}_i)_i$ form an equilibrium of the hat economy. Clearly $\hat{x} = \hat{y} - e$, and so it remains to be shown that $\hat{x}_i \in X^*_i(p)$ and $\hat{y}_j \in Y^*_j(p)$.

Suppose $\hat{x}_i \not\in X^*_i(p)$, so that there is some $x_i \in X_i$ such that $x_i \succ_i \hat{x}_i$ and $p \cdot x_i \leq w_i$. Since $\hat{x}_i$ is in the interior of K, it follows from the assumption that X_i is convex that for λ small enough $x'_i := \lambda x_i + (1-\lambda)\hat{x}_i$ is in $\hat{X}_i$. But then $x'_i \succ_i \hat{x}$, since $\preceq_i$ is convex*, while clearly $p \cdot x'_i \leq w_i$. So $\hat{x}_i \not\in \hat{X}^*_i(p)$.

Finally, suppose that $\hat{y}_j \not\in Y^*_j(p)$. By an analogous argument there is some $y'_j \in \hat{Y}_j$ such that $p \cdot y'_j > p \cdot \hat{y}_j$, and so $\hat{y}_j \not\in \hat{Y}^*_j(p)$.

□

8 Kakutani's Theorem and Debreu's Theorem

Let A,B be subsets of $\mathbb{R}^n$,$\mathbb{R}^m$. A *correspondence* $\Gamma\colon A \to B$ is a map that assigns to each $a \in A$ a subset $\Gamma(a) \subseteq B$. We say that Γ is nonempty / closed / convex if each $\Gamma(a)$ is nonempty / closed / convex.

A correspondence $\Gamma\colon A \to B$ is *upper hemicontinuous* at a if for all $a_n \to a$ and $b_n \to b$ such that $b_n \in \Gamma(a_n)$ it holds that $b \in \Gamma(a)$. Note that this implies that $\Gamma(a)$ is closed for every a, by taking $a_n = a$. It is said to be upper-hemicontinuous if it is upper-hemicontinuous at all $a \in A$.

Claim 8.1. *A correspondence $\Gamma\colon A \to B$ is upper hemicontinuous iff its graph $\{(a,b) : b \in \Gamma(a)\}$ is a closed subset of $A \times B$.*

Proof. Suppose Γ is upper hemicontinuous, and consider a converging sequence $(a_n,b_n) \to (a,b)$ in its graph. Then upper hemicontinuity implies that (a,b) is also in the graph. Hence the graph is closed.

Conversely, suppose the graph is closed, and consider $a_n \to a$ and $b_n \to b$ such that $b_n \in \Gamma(a_n)$. Since the graph is closed, (a,b) is in the graph, i.e., $b \in \Gamma(a)$. □

A *fixed point* of a correspondence $\Gamma\colon A \to A$ is $a \in A$ such that $a \in \Gamma(a)$.

Theorem 8.2 (Kakutani)**.** *Let A be a compact convex subset of $\mathbb{R}^n$, and let $\Gamma\colon A \to A$ be a nonempty, upper-hemicontinuous, convex correspondence. Then Γ has a fixed point.*

Denote $\Delta^L = \{p \in \mathbb{R}^L : \sum_\ell p_\ell = 1\}$. In the next claim we "forget" all we know about Z^*, and only assume what is explicitly written.

Theorem 8.3 (Debreu)**.** *Let C be a compact convex subset of $\mathbb{R}^L$. Suppose that a correspondence $Z^* : \Delta^L \to C$ is nonempty, upper-hemicontinuous, convex, and satisfies $p \cdot z \le 0$ for all $z \in Z^*(p)$. Then there is a $p \in \Delta^L$ and a $z \in Z^*(p)$ such that $z \in -\mathbb{R}^L$.*

Proof. Define the correspondence $M\colon C \to \Delta^L$ by $M(z) = \operatorname{argmax}_p p \cdot z$. That is, $M(z)$ is the set of p that maximize $p \cdot z$. It is easy to see that M is upper-hemicontinuous, non-empty and convex.

Let $\Gamma\colon \Delta^L \times C \to \Delta^L \times C$ be the correspondence given by $\Gamma(p,z) = M(z) \times Z^*(p)$. It is again easy to see that the domain $\Delta^L \times C$ is compact and convex and that Γ is nonempty, upper-hemicontinuous and convex. Thus, by Kakutani's Theorem, it has a fixed point (p,z). That is, there are p and z such that

$$p \in M(z) \quad \text{and} \quad z \in Z^*(p).$$

Since $p \in M(z)$, $p' \cdot z \le p \cdot z$ for any $p' \in \Delta^L$. And since $z \in Z^*(p)$, $p \cdot z \le 0$. Hence $p' \cdot z \le 0$ for any $p \in \Delta^L$, and so $z \in \mathbb{R}^L_-$. □

9 Existence of equilibria

To prove the existence of equilibria we will use Theorem 8.3. It requires compactness; that will be provided by Claim 7.1 and Proposition 7.2. It also requires upper-hemicontinuity.

Proposition 9.1. *If Y is compact then the correspondence Y^* is upper-hemicontinuous.*

Suppose furthermore that Y is convex, each X_i is compact and convex, each $\succeq_i$ is convex and LNS, and e_i is in the interior of X_i. Then the correspondence Z^* is nonempty, upper-semicontinuous and convex.*

Theorem 9.2. *Suppose each X_i is a closed, convex subset of $\mathbb{R}^L_+$, each $\succeq_i$ is convex* and LNS, each e_i is in the interior of X_i, $Y \cap \mathbb{R}^L_+ = \{0\}$, $Y \cap (-Y) = \{0\}$, and Y is convex. Then there exists an equilibrium.*

Proof. By Claim 7.1 A is bounded. Hence by Proposition 7.2 any equilibrium of the hat economy is an equilibrium of the original economy. We can thus assume henceforth that each X_i is compact, as is Y. By Proposition 9.1 the excess demand correspondence Z^* is upper-hemicontinuous and convex. By Claim 6.1 it satisfies Walras's Law. We can therefore apply Theorem 8.3 to conclude that there is some $z \in R^L_-$ such that $z \in Z^*(p)$. Thus, by Claim 6.3 there exists an equilibrium. □

10 Approximate equilibria

In this section we consider exchange economies, with I consumers and L goods, as usual. We will start by proving an existence theorem for equilibria, under strong conditions.

Theorem 10.1. *Consider an exchange economy with each $X_i = [0,x]^L$ for some x, and with strictly monotone and convex preferences. Suppose also that each e_i is in the interior of X_i. Then there exists an equilibrium with prices $p \gg 0$.*

Proof. Recall that we denote $\Delta^L = \{p \in \mathbb{R}^L_+ : \sum_\ell p_\ell = 1\}$. For each $p \in \Delta^L$, the set of feasible consumption bundles $\{x_i \in X_i : p \cdot x_i \leq p \cdot e_i\}$ is compact and non-empty. Since preferences are convex and X_i is compact, $X_i^*(p) = X_i^*(p, p \cdot w)$ is compact, convex, and non-empty, and so is the excess demand $Z^*(p) = \sum_i X_i^*(p) - \sum_i e_i$, which is furthermore upper-hemicontinuous since each e_i is internal (see Proposition 9.1). Since preferences are strictly monotone they are LNS, and thus Walras's Law implies that $p \cdot z = 0$ for all $z \in Z^*(p)$. By Theorem 8.3 there is a p such that $Z^*(p) \cap \mathbb{R}^L_-$ is non-empty. Note that it must be that $p \gg 0$, since preferences are strictly monotone, and thus if $p_\ell = 0$ for some commodity ℓ then $X_i^*(p)$ will only include elements in which the demand for good ℓ is equal to x (i.e., the maximal possible), but then, since e_i is internal, the demand for ℓ will exceed supply, and it would be impossible that $Z^*(p)$ includes elements of $\mathbb{R}^L_-$.

It follows that there are $(x_i)_i$ such that $x_i \in X_i^*(p)$, and $\sum_i (x_i - e_i) \in \mathbb{R}^L_-$. To finish the proof we show that this sum vanishes. Assume it does not. Then, since $p \gg 0$, $p \cdot \sum_i (x_i - e_i) < 0$. But by LNS we know that $p \cdot x_i = p \cdot e_i$, and thus we have reached a contradiction. □

An *approximate equilibrium* consists of a price vector p together with consumption vectors $(x_i)_i$ such that

1. For all i, $p \cdot x_i \leq p \cdot e_i$.

2. For all i except at most L, $x_i \in X_i^*(p, p \cdot e_i)$.

3. $(x_i)_i$ is an allocation: $\sum_i x_i = \sum_i e_i$.

Theorem 10.2. *Consider an exchange economy with each $X_i = [0,x]^L$ for some x, and with strictly monotone and convex preferences. Suppose also that each e_i is in the interior of X_i. Then there exists an approximate equilibrium with prices $p \gg 0$.*

The only difference between this Theorem and Theorem 10.1 is that there is no assumption that preferences are convex. This implies that there will not always be an equilibrium.

Proof of Theorem 10.2. By our assumptions Z^* is a non-empty, compact correspondence, which is furthermore upper-hemicontinuous (see Proposition 9.1). However, since we have not assumed that preferences are convex, there will not necessarily be an equilibrium. I.e., there is not necessarily a price vector p such that $0 \in Z^*(p)$. Denote by $Z_c^*(p)$ the convex hull of $Z^*(p)$. Then clearly Z_c^* is a non-empty, compact, convex, upper-hemicontinuous correspondence. Thus, as in the proof of Theorem 10.1, there must exist a $p \gg 0$ such that $0 \in Z_c^*(p)$.

To finish the proof, we will need the following result, which we will prove later. We denote by $\mathrm{Conv}(A)$ the convex hull of a set A.

Lemma 10.3 (Shapley-Folkman). *Let $A_1,\dots,A_K$ be subsets of $\mathbb{R}^L$, and suppose that $x \in \mathrm{Conv}(A_1+\cdots+A_K)$. Then there exist $a_1,\dots,a_k$ such that*

1. *$a_i \in \mathrm{Conv}(A_i)$ for all i,*
2. *$a_i \in A_i$ for all but L values of i,*
3. *$x = a_1+\cdots+a_K$.*

Applying the lemma to $0 \in Z_c^*(p) = \sum_i X_i^*(p) - e_i$, we can conclude that there are $(z_i)_i$ such that

1. $z_i \in \mathrm{Conv}(X_i^*(p) - e_i)$.
2. $z_i \in X_i^*(p) - e_i$ for all but L consumers i.
3. $0 = \sum_i z_i$.

Denoting $x_i = z_i + e_i$, this implies that

1. $x_i \in \mathrm{Conv}(X_i^*(p))$, and hence $p \cdot x_i \le p \cdot e_i$.
2. $x_i \in X_i^*(p)$ for all but L consumers i.
3. $\sum_i x_i = \sum_i e_i$.

Thus $(x_i)_i$ and p form an approximate equilibrium. □

We now turn to prove Lemma 10.3. This result is in fact a generalization of the Carathéodory Theorem for convex hulls.

Theorem 10.4 (Carathéodory). *Suppose $a \in \mathrm{Conv}(A) \subseteq \mathbb{R}^L$. Then a is a convex combination of at most $L+1$ elements of A.*

Proof. Suppose that a is the convex combination $\sum_\ell \lambda_\ell a_\ell$ of $\{a_1,\dots,a_n\} \subseteq A$, and that n is the minimal such that is possible. In particular, this means that $\lambda_\ell > 0$.

Define the linear map $\Phi\colon \mathbb{R}^n \to \mathbb{R}^L \times \mathbb{R}$ by

$$\Phi(x_1,\dots,x_n) = \left(\sum_\ell x_\ell a_\ell, \sum_\ell x_\ell\right).$$

The kernel of Φ (i.e., $\{x \in \mathbb{R}^n : \Phi(x) = (0,0)\}$) has dimension at least $n-(L+1)$. Assume towards a contradiction that $n > L+1$. Then this dimension is positive, and there is some nonzero $x \in \mathbb{R}^K$ such that $\sum_\ell x_\ell a_\ell = 0 = \sum_\ell x_\ell$. By multiplying x by a constant we can furthermore require that $(x+\lambda)_\ell \geq 0$ for all i (since $\lambda >> 0$) and that $(x+\lambda)_\ell = 0$ for some i. Denote $\eta = x+\lambda$. Then

$$\sum_\ell \eta_\ell a_\ell = \sum_\ell x_\ell a_\ell + \sum_\ell \lambda_\ell a_\ell = a$$

and

$$\sum_\ell \eta_\ell = \sum_\ell x_\ell + \sum_\ell \lambda_\ell = 1,$$

so that a is a convex combination of less than n elements of A, and we have reached a contradiction. □

Proof of Lemma 10.3. We note first that $\mathrm{Conv}(A_1 + \cdots + A_K) = \mathrm{Conv}(A_1) + \cdots + \mathrm{Conv}(A_K)$. Thus $x = \sum_i a_i$ where $a_i \in \mathrm{Conv}(A_i)$. Suppose each a_i is a convex combination $a_i = \lambda_{i,1}a_{i,1} + \cdots + \lambda_{i,n_i}a_{i,n_i}$ of elements of A_i, and that this representation minimizes the total number of coefficients $n = \sum_i n_i$. This means that $\lambda_{i,j} > 0$ for all i,j.

Define the linear map $\Phi\colon \mathbb{R}^{n_1} \times \cdots \times \mathbb{R}^{n_K} \to \mathbb{R}^L \times \mathbb{R}^K$ by

$$\Phi(x_1,\ldots,x_K) = \left(\sum_i \sum_{\ell=1}^{n_i} x_{i,\ell}a_{i,\ell}, \sum_\ell x_{1,\ell}, \ldots, \sum_\ell x_{K,\ell}\right).$$

The kernel of Φ has dimension at least $n-(L+K)$. By an argument analogous to the one in Theorem 10.4 we conclude that $n \leq L+K$, as $n > L+K$ implies that there is some $(x_1,\ldots,x_K)$ such that

$$\sum_i \sum_{\ell=1}^{n_i} x_{i,\ell}a_{i,\ell} = 0 = \sum_\ell x_{1,\ell} = \cdots = \sum_\ell x_{K,\ell},$$

which, after multiplication by a constant can be added to λ to yield convex combinations with a combined smaller support.

Finally, it follows from $n \leq L+K$ and the pigeon hole principle that there are at least $K-L$ values of i such that $n_i = 1$, and hence, for these i, $a_i \in A_i$. □

A nice related theorem is the following.

Theorem 10.5 (Kirchberger)**.** *Let A ("sheep") and B ("wolves") be finite subsets of $\mathbb{R}^2$. Suppose that for every $C \subseteq A \cup B$ of size 4 there is a line that strictly separates $A \cap C$ from $B \cap C$ (i.e., the sheep in C can be separated from the wolves in C by a straight fence). Then A can be strictly separated by a line from B (the sheep can be separated from the wolves by a straight fence).*

11 Scitovsky contours

Consider an exchange economy with each $\succeq_i$ convex. The *Scitovsky contour* of $(x_i)_i$ is

$$S(x_1,\ldots,x_I) = \sum_i \{x_i' : x_i' \succeq_i x_i\}.$$

That is S is the set of all total consumptions that can be decomposed into individual consumptions that are at least as good as $(x_i)_i$.

Suppose $(x_i)_i$ and p form an equilibrium, so that $p \cdot x' \geq p \cdot x$ for every $x' \in S(x_1,\ldots,x_I)$. It follows that if $p \cdot \hat{x} < p \cdot x$ then, for any $(\hat{x}_i)_i$ such that $\hat{x} = \sum_i \hat{x}_i$, there must be some consumer i for whom $x_i \succ_i \hat{x}_i$.

The lesson is that if current total consumption is x and prices are at p, then an alternative total consumption $\hat{x}$ can be ruled out (on the grounds of making someone worse off) just by considering the value of this consumption in terms of p, without any knowledge of the preferences.

12 The core

Given vectors $a, b \in \mathbb{R}^n$, we denote $a \geq b$ if $a_i \geq b_i$ for all coordinates i, we denote $a > b$ if $a \geq b$ and $a \neq b$, and denote $a \gg b$ if $a_i > b_i$ for all i.

In this section we will consider an exchange economy in which $X_i = \mathbb{R}^+_L$, $e_i \gg 0$, and $\succeq_i$ is closed, strictly convex and *strictly monotone*: if $x'_i > x_i$ then $x'_i \succ_i x_i$. Note that this implies that preferences are LNS.

A *coalition* is a subset S of the consumers. A partial allocation is $(x'_i)_{i \in S}$ such that each $x_i \in X_i$ and $\sum_{i \in S} x'_i = \sum_{i \in S} e_i$.

Consider an allocation $(x_i)_i$. We say that a coalition S *blocks* $(x_i)_i$ if there is a partial allocation $(x'_i)_{i \in S}$ such that $x'_i \succ_i x_i$ for all $i \in S$. We say that a coalition S *weakly blocks* $(x_i)_i$ if there is a partial allocation $(x'_i)_{i \in S}$ such that $x'_i \succeq_i x_i$ for all $i \in S$, and $x'_\ell \succ_\ell x_\ell$ for some $\ell \in S$.

Claim 12.1. *S blocks $(x_i)_i$ iff it weakly blocks it.*

Proof. Clearly blocking implies weak blocking. For the other direction, assume $(x'_i)_{i \in S}$ witnesses that S weakly blocks $(x_i)_i$, and that $x'_\ell \succ_\ell x_\ell$. Since preferences are closed, for $\varepsilon > 0$ small enough it holds that $(1-\varepsilon)x'_\ell \succ_\ell x_\ell$.

Let $\bar{x}_\ell = (1-\varepsilon)x'_\ell$, and for the rest of the $i \in S$ let $\bar{x}_i = x'_i + \frac{\varepsilon}{|S|-1}x'_\ell$. Then $\sum_{i \in S} \bar{x}_i = \sum_{i \in S} x'_i$, and so $\bar{x}_i$ is an allocation. Furthermore, by strict monotonicity, $\bar{x}_i \succ_i \bar{x}_i$, and so S blocks $(x_i)_i$. □

The *core* of the economy is the set of allocations that are not blocked by any coalition.

Claim 12.2. *Suppose $(x_i)_i$ and p form an equilibrium. Then $(x_i)_i$ is in the core.*

Proof. Suppose not, so that the partial allocation $(x'_i)_{i \in S}$ witnesses that S blocks $(x_i)_i$. Since $x'_i \succ_i x_i$, it follows from the equilibrium condition that $p \cdot x'_i > p \cdot x_i$. Since preferences are LNS, $p \cdot x_i = p \cdot e_i$ (Lemma 2.1) and so $p \cdot x'_i > p \cdot e_i$. Thus

$$\sum_{i \in S} p \cdot x'_i > \sum_{i \in S} p \cdot e_i,$$

in contradiction to the partial allocation condition $\sum_{i \in S} x'_i = \sum_{i \in S} e_i$. □

In general, not every element of the core belongs to an equilibrium. For example, in a two consumer economy, every allocation $(x_i)_i$ that is Pareto optimal (i.e., not blocked by the coalition of both consumers) and satisfies $x_i \succeq_i e_i$ (i.e., not blocked by a single consumer) is in the core.

The N^{th} replica economy has $N \cdot I$ consumers, with each consumer of the original economy duplicated N times. Each duplicate has the same endowment and preference as the original. We index consumers by (i, n), where $i \in \{1, \ldots, I\}$ and $n \in \{1, \ldots, N\}$. Thus $e_{i,n} = e_i$ and $\succeq_{i,n} = \succeq_i$.

Proposition 12.3. *Let $(x_{i,n})_{i,n}$ be in the core of the N^{th} replica economy. Then $x_{i,n} = x_{i,n'}$ for all i, n, n'.*

Proof. For each i, fix $n(i)$ such that $x_{i,n(i)}$ is $\succeq_i$-minimal. Let $S = \{(i,n(i)) : i \in \{1,\ldots,I\}\}$. For $i \in S$, let

$$x'_i = \frac{1}{N}\sum_{n=1}^{N} x_{i,n},$$

and let $x'_{i,n} = x'_i$.

Suppose that $x_{\ell,n} \neq x_{\ell,n'}$ for some ℓ, n, n'. Note that $x'_{i,n(i)} \succeq_i x_{i,n(i)}$, with strict inequality for $i = \ell$, by strict convexity. Thus, to prove that S blocks $(x_i)_i$, it remains to be shown that $(x'_i)_{i\in S}$ is a partial allocation:

$$\sum_{(i,n(i))\in S} x'_{i,n(i)} = \sum_i x'_i = \sum_i \frac{1}{N}\sum_n x_{i,n} = \frac{1}{N}\sum_i\sum_n x_{i,n} = \frac{1}{N}\sum_{i,n} e_i = \sum_i e_i = \sum_{(i,n(i))\in S} e_i.$$

□

Given Proposition 12.3, we can identify every element of the core of the replica economy with an allocation of the original economy. Indeed, this allocation will be in the core of the original economy. More generally, the core of the N^{th} replica economy will contain the core of the $N+1^{\text{th}}$ replica economy.

Theorem 12.4 (Debreu-Scarf)**.** *Suppose that $(x_i)_i$ is in the core of the N^{th} replica economy for all N. Then there exists a price vector p such that $(x_i)_i$ and p form a Walrasian equilibrium.*

Proof. Let $(x_i)_i$ be in the core of the N^{th} replica economy for all N. Let $P_i = \{z_i : e_i + z_i \succ_i x_i\}$. Let P be the convex hull of $\cup_i P_i$. I.e., P is the set of all z such that there exist $\alpha_i \geq 0$, $\sum_i \alpha_i = 1$ and $z_i \in P_i$ with $\sum_i \alpha_i z_i = z$. Since P is open $\cup_i P_i$ is open, and P is open.

We claim that P does not contain 0. Suppose it does. Then, since P is open, it also contains some $z \in \mathbb{R}^L_-$. Hence there are z_i and α_i such that z is equal to the convex combination $\sum_i \alpha_i z_i$. Since each P_i is open, we can further assume that each α_i is rational, so that $\alpha_i = k_i/N$ for some k_i and common denominator N. Note that $\sum_i k_i = N$.

Consider the N^{th} replica economy, and a coalition S of size N that consists of k_i consumers of each type i. For $(i,n) \in S$, let $x'_{i,n} = e_i + z_i - z$. Since $z \in \mathbb{R}^L_-$, by monotonicity we have that $x'_{i,n} \succ e_i + z_i$. Since $z_i \in P_i$, $e_i + z_i \succ_i x_i$, and so $x'_{i,n} \succ_i x_i$. To show that S blocks $(x_i)_i$, it remains to be shown that $(x'_{i,n})_{i,n}$ is a partial allocation:

$$\sum_{(i,n)\in S} x'_{i,n} = \sum_i k_i(e_i + z_i - z) = \sum_i k_i e_i + N\sum_i \frac{k_i}{N} z_i - Nz = \sum_i k_i e_i = \sum_{(i,n)\in S} e_{i,n}.$$

We thus conclude that $0 \notin P$.

By the there is some (nonzero) price vector $p \in \mathbb{R}^L$ and a $w \in \mathbb{R}$ such that $p \cdot (e+y) \leq w$ for all $e + y \in \hat{Y}$ and such that $p \cdot x^+ > w$ for all $x^+ \in X^+$. Fix some such p. Note that $x \in \hat{Y}$, since $((x_i)_i, (y_j)_j)$ is an allocation. Hence $p \cdot x \leq w$. On the other hand, by LNS, x is in the closure of X^+ and so $p \cdot x \geq w$. We thus have that $w = p \cdot x$. Since $((x_i)_i, (y_j)_j)$ is an allocation, $w = p \cdot (e+y)$.

We claim first that $y_j \in Y_j^*(p)$ for all j. By Claim 3.1, this is equivalent to $p \cdot y' \leq p \cdot y$ for any $y' \in \sum_j Y_j$. This in turn is equivalent to $p \cdot (e + y') \leq p \cdot (e + y) = w$, which we have already shown above.

Let

$$t_i = p \cdot x_i - (p \cdot e_i + \sum_j \theta_{ij} p \cdot y_j).$$

Then

$$\sum t_i = p \cdot x - (p \cdot e + \sum_i \sum_j \theta_{ij} p \cdot y_j) = p \cdot x - (p \cdot e + \sum_j p \cdot y_j) = 0,$$

where the first equality is a substitution of the definitions of x and e, the second a consequence of $\sum_i \theta_{ij} = 1$, and the third follows because $((x_i)_i, (y_j)_j)$ is an allocation.

By our definition of t_i, $p \cdot x_i = w_i$. To show that we have constructed a quasi-equilibrium we show that if $x_i' \succ_i x_i$ then $p \cdot x_i' \geq w_i$. Suppose towards a contradiction that there is an $x_\ell^+ \in X_\ell$ such that $x_\ell^+ \succ_\ell x_\ell$ and $p \cdot x_\ell^+ < w_\ell - \varepsilon$ for some $\varepsilon > 0$. By LNS, we can find for each $i \neq \ell$ an $x_i^+ \in X_i$ such that $x_i^+ \succ_i x_i$ and $p \cdot x_i^+ < w_i + \varepsilon / I$. It follows that

$$p \cdot \sum_i x_i^+ < p \cdot \sum_i x_i.$$

Denote $x^+ = \sum_i x_i^+$. Then $x^+ \in X^+$, and yet $p \cdot x^+ < p \cdot x = p \cdot e$, and so we have reached a contradiction. Thus we have constructed a quasi-equilibrium.

Finally, suppose that each x_i is an interior point. Then for each x_i there is an $\hat{x}_i$ such that $p \cdot \hat{x}_i < w_i$. It thus follows from Claim 2.2 that $x_i \in X^*(p, w_i)$, and so we have an equilibrium.

□

13 The core via approximate equilibria

Consider an exchange economy, an allocation $(x_i)_i$ and a price vector p. To quantify how far $(x_i)_i$ and p are from forming an equilibrium, we consider two quantities. First, we denote by η_i the amount by which i exceeds her budget:

$$\eta_i = \max\{p \cdot (x_i - e_i), 0\},$$

and by $\eta = \frac{1}{I}\sum_i \eta_i$ the average amount by which the consumers exceed their budgets. Note that $\eta = 0$ if and only if each consumer satisfy her budget contraint.

In equilibrium, it is impossible that a consumer chooses x_i, but strictly prefers some affordable x_i' to x_i. Here by "affordable" we mean $p \cdot (e_i - x_i') \geq 0$. We denote by ζ_i the amount of money that i could save (relative to the worth of her endowment) while improving her consumption:

$$\zeta_i = \sup_{x_i' \succ_i x_i} \max\left\{p \cdot (e_i - x_i'), 0\right\},$$

and by $\zeta = \frac{1}{I}\zeta_i$ the average of these amounts.

Note that an allocation $(x_i)_i$ is an equilibrium if and only if $\eta_i = 0$ and $\zeta_i = 0$. Thus η_i and ζ_i quantify the extent by which consumer i violates the equilibrium conditions

Theorem 13.1. *Consider an exchange economy with $X_i = \mathbb{R}^L_+$, strictly monotone preferences, and $\sum_i e_i \gg 0$. Denote*

$$M = L \cdot \max_{i,\ell} e_{i,\ell}.$$

For every $(x_i)_i$ in the core there exists a $p \in \Delta^L$ such that $\eta \leq \frac{M}{I}$ and $\zeta \leq \frac{M}{I}$.

To prove this theorem, let $(x_i)_i$ be in the core. Let

$$B_i = \{x_i' - e_i : x_i' \succ_i x_i\} \cup \{0\},$$

and let $B = \sum_i B_i$.

Lemma 13.2. $B \cap \mathbb{R}^L_- = \{0\}$.

Proof. Fix $b \sum_i b_i \in B$, with $b_i \in B_i$. Let S denote those consumers for which $b_i \neq 0$. For the rest $b_i = 0$, and so $b = \sum_{i \in S} b_i$. By the definition of B_i, $b = \sum_{i \in S} x_i' - e_i$, for some $x_i' \succ_i x_i$.

Since x is in the core, S cannot be a blocking coalition, and thus either S is empty, or else $(x_i')_{i \in S}$ is not a partial allocation. In the former case $b = 0$, and we are done. In the latter case, $b \neq 0$. Suppose towards a contradiction that $b \in \mathbb{R}^L_-$. Let $x_k'' = x_k' - b$ for some $k \in S$, for for all other $i \in S$ let $x_i'' = x_i'$. Then $\sum_{i \in S} x_i'' = -b + \sum_{i \in S} x_i' = 0$, and $(x_i'')_{i \in S}$ is a partial allocation. Since preferences are monotone, $x_i'' \succ_i x_i$, and so S is a blocking coalition, and we have reached a contradiction. □

The next lemma shows that a slightly weaker statement holds for the convex hull of B. Denote $m = (M, M, \dots, M) \in \mathbb{R}^L$.

Lemma 13.3. $\mathrm{Conv}(B+m) \cap \mathbb{R}^L_- \subseteq \{0\}$.

Proof. Fix $b = \sum_i b_i \in B$. By the Shapley-Folkman Lemma (Lemma 10.3), there is a subset of the consumers S, of size at least $I - L$, such that $b_i \in B_i$ for $i \in S$. For $i \not\in S$, we claim that $b_i \geq -m/L$. This clearly holds if $b_i = 0$. Otherwise, b_i is the convex hull of points of the form $x'_i - e_i \geq x'_i - m/L \geq -m/L$, and thus again $b_i \geq -m/L$.

Recalling that there are at most L many indices i not in S, $\sum_{i \not\in S} b_i \geq -m$. Thus

$$b + m = m + \sum_i b_i = m + \sum_{i \in S} b_i + \sum_{i \not\in S} b_i \geq m + \sum_{i \in S} b_i - m = \sum_{i \in S} b_i.$$

By Lemma 13.2, $\sum_{i \in S} b_i = 0$ if it is in R^L_-, and so we have proved the claim. □

We now apply the separating hyperplane theorem, which guarantees the existence of a $p \neq 0 \in \mathbb{R}^L$ such that $p \cdot z \leq 0$ for $z \in \mathbb{R}^L_-$, and $p \cdot z \geq 0$ for $z \in \mathrm{Conv}(B+m)$. From the former condition it follows that $p \geq 0$, and so, by rescaling, we can assume that $p \in \Delta^L$. It thus follows from the latter condition that for each $b \in B$

$$p \cdot b \geq -p \cdot m = -M.$$

Let $\bar{B}_i$ denote the closure of B_i, and note that $x_i - e_i \in \bar{B}_i$, since by the monotonicity of the preferences $x_i + (\varepsilon, \dots, \varepsilon) \succ_i x_i$. Let $\bar{B} = \sum_i \bar{B}_i$. Clearly, $p \cdot z \geq 0$ for $z \in \mathrm{Conv}(\bar{B} + m)$, and in particular

$$p \cdot b \geq -M$$

for every $b \in \bar{B}$.

Denote by S the set of consumers that underspend, i.e., those i for which $p \cdot (x_i - e_i) < 0$. Then $\sum_{i \in S} x_i - e_i \in \bar{B}$ (as 0 is in each B_i), and so

$$\sum_{i \in S} p \cdot (x_i - e_i) \geq -M.$$

I.e., the total amount underspent by those who underspend is at most M. Recalling that $(x_i)_i$ is an allocation, $\sum_i x_i - e_i = 0$, and so on average consumers do not overspend: $\sum_i p \cdot (x_i - e_i) = 0$. Thus the total amount overspent by those who overspend is at most M:

$$\sum_i \max\{p \cdot (x_i - e_i), 0\} \leq M.$$

Dividing both sides by I yields $\eta \leq \frac{M}{I}$.

It remains to be shown that $\zeta \leq \frac{M}{I}$. Let T be the set of consumers for whom $\sup_{x'_i \succ_i x_i} p \cdot (e_i - x'_i) \geq 0$. Note that

$$\zeta = \frac{1}{I} \sum_i \sup_{x'_i \succ_i x_i} \max\left\{p \cdot (e_i - x'_i), 0\right\} = \frac{1}{I} \sum_{i \in T} \sup_{x'_i \succ_i x_i} p \cdot (e_i - x'_i).$$

Since $p \cdot b \geq -M$ for every $b \in B$, we have that if $x_i' \succ_i x_i$ for each $i \in T$, then

$$\frac{1}{I} p \cdot \sum_{i \in T} (x_i' - e_i) \geq -\frac{M}{I},$$

and

$$\frac{1}{I} \sum_{i \in T} p \cdot (e_i - x_i') \leq \frac{M}{I}.$$

Hence

$$\zeta = \frac{1}{I} \sum_{i \in T} \sup_{x_i' \succ_i x_i} p \cdot (e_i - x_i') \leq \frac{M}{I}.$$

14 Partial equilibrium: consumers

Consider an private ownership economy with two commodities: commodity 1 and 2, where 2 shall be referred to as "money", or the "numeraire". We will denote a consumption bundle of agent i by a pair (x_i, m_i), where x_i is the consumption of good 1 and m_i is the consumption of money. We assume that agents have quasilinear utilities, so that the preference of agent i is represented by

$$u_i(x_i, m_i) = u_i(x_i) + m_i,$$

and furthermore that $u_i(x_i) \in C^2$ is strictly concave and strictly increasing, and that $u_i(0) = 0$. We can think here of $u_i(x_i)$ as the utility for consuming x_i of commodity 1, measured in units of money.

The consumption set is $X_i = \mathbb{R}_+ \times \mathbb{R}$, so that commodity 1 can only be consumed in positive amounts, but money can be consumed in both negative or positive amounts. The latter is an important assumption. We normalize the price of money to 1, so that prices can be specified by a single number $p \in \mathbb{R}$, the price of the first commodity.

Consider the consumer's problem. Given price p and wealth w_i, she will choose to consume (x_i^*, m_i^*) if the numbers maximize $u_i(x_i^*) + m_i^*$ subject to the constraint $px_i^* + m_i \leq w_i$. We solve this problem in two steps. First given a choice of x_i^*, the unique optimal choice of m_i^* is clearly $m_i^* = w_i - p \cdot x_i^*$. Hence

$$x_i^* \in \operatorname*{argmax}_x u_i(x_i) + w_i - p \cdot x_i.$$

The solution to this problem is independent of w_i, and is hence

$$x_i^* \in \operatorname*{argmax}_x u_i(x_i) - p \cdot x_i.$$

We can thus write $x_i^*(p, w_i) = x_i^*(p)$. The demand for money is

$$m_i^*(p, w_i) = w_i - p \cdot x_i^*(p).$$

By our assumption that u_i is strictly increasing and in C^2, if $x_i^* = 0$ then $u_i'(x_i^*) \leq p$, and the same holds with equality if if $x_i^* > 0$. In this case we have that $u_i'(x_i^*(p)) = p$. I.e., $x_i^*(p)$ is the inverse of the u_i' since u_i is strictly concave, there is a unique solution. Thus, unless prices are too high and the agent prefers not to consume at all, the demand for commodity 1 is chosen so that the marginal utility of the commodity matches the price.

The indirect utility of consumer i for price p and wealth w_i is

$$\begin{aligned} v_i(p, w_i) &= u_i(x_i^*(p)) + m_i^*(p, w_i) \\ &= u_i(x_i^*(p)) + w_i - p \cdot x_i^*(p) \\ &= w_i + u_i(x_i^*(p)) - p \cdot x_i^*(p) \end{aligned}$$

The term

$$CS_i(p) = u_i(x_i^*(p)) - p \cdot x_i^*(p)$$

is called the *consumer surplus*.

For p such that $x_i^*(p) > 0$, we know that $x_i^*(p)$ is the inverse of $u_i'(x_i)$. Denote $\bar{p} \in \mathbb{R} \cup \{\infty\}$ the lowest p such that $x_i^*(p) = 0$.

Recall that if $F \in C^2$ is strictly increasing and f is its derivative, then

$$\int_1^b f^{-1}(y)\,\mathrm{d}y = [yf^{-1}(y) - F(f^{-1}(y))]_a^b.$$

Thus

$$\int_p^{\bar{p}} x_i^*(q)\,\mathrm{d}q = [qx_i^*(q) - u_i(x_i^*(q))]_p^{\bar{p}} = u_i(x_i^*(p)) - px_i^*(p) = CS_i(p).$$

In case $\bar{p} = \infty$ to show that the second equality holds we need to show that $\lim_{q\to\infty} qx_i^*(q) = 0$. This is left as an exercise.

Note also that

$$CS_i = \int_0^{x_i^*(p)} [u_i'(x) - p]\,\mathrm{d}x.$$

The integrand $u_i'(x) - p$ is the marginal benefit to the consumer for consuming at level x, where she gains marginal utility $u_i'(p)$ and pays marginal cost p. This integrand is positive at all $x < x_i^*(p)$ since $u_i'(x) < p$ in that region.

Let $x^*(p) = \sum_i x_i^*(p)$. If we let u' be the inverse of x, and u the integral of u', then u is a utility function that induces demand $x^*(p)$. We thus always have a representative consumer. A simple calculation shows that the consumer surplus for this representative consumer is equal to the sums of the surpluses of the individual consumers.

15 Partial equilibrium: production

Suppose that it costs $c_i(y)$ for producer i to produce y units of commodity 1. This corresponds to a firm whose production set is $Y = \{(y, -m) : y \in \mathbb{R}_+, m \in \mathbb{R}_+, m \geq c(y)\}$. We will assume that $c : \mathbb{R}_+ \to \mathbb{R}_+$ is in C^2, strictly increasing and strictly convex. Note that (strict) convexity of Y corresponds to (strict) convexity of c.

At price p, firm i will choose to produce $y_i^*(p)$ units of commodity 1, which will cost it $-c_i(y)$ money. The firm's problem is hence solved by

$$y_i^*(p) \in \underset{y \in Y}{\operatorname{argmax}}\, p \cdot y - c_i(y)$$

If $y_i^* = 0$ then $c'(y^*) \geq p$. If $y_i^* > 0$ then it must be that $c_i'(y^*(p)) = p$, and thus $y_i^*(p)$ is the inverse of $c_i'(y)$.

We can let $Y = \sum_i Y_i$ and $y^*(p) = \sum_i y_i^*(p)$. By our assumptions, $y^*(p)$ is strictly increasing. Let $c(y) = \min\{m : (y, -m) \in Y\}$. The set Y is strictly convex, since it is the sum of strictly convex sets. Hence $c(y)$ is strictly convex.

16 Partial equilibrium conditions

We will assume that consumers are endowed with money only. We will consider equilibria in which $x_i > 0$ for all i. Thus, an allocation $((x_i, m_i)_i, (y_i)_i)$ and a price vector p such that $x_i > 0$ are an equilibrium if $u_i'(x_i) = p$, $c_i'(y_i) = p$, $\sum_i x_i = \sum_i y_i$ and $\sum_i m_i = \sum_i e_i$. Note that it must be that $p > 0$, since otherwise the consumers will consume an infinite amount.

The next claim shows that we can dispense with the last condition.

Claim 16.1. *In a general private ownership economy with LNS preferences, an allocation $((x_i)_i, (y_i)_i)$ and a price vector p with $p_L \neq 0$ are an equilibrium if $x_i \in X_i^*(p, w_i)$, $y_j \in Y_j^*(p)$, and $\sum_i x_{i,\ell} = \sum_j y_{j,\ell}$ for $\ell \in \{1, \dots, L-1\}$.*

Proof. Since preferences are LNS, by Lemma 2.1 $p \cdot x_i = w_i$. Summing over i yields

$$p \cdot \sum_i x_i - w_i = 0.$$

Writing this expression commodity by commodity and separating the last commodity yields

$$p_L \cdot \sum_i (x_{i,L} - w_{i,L}) = -\sum_{\ell \neq L} p_\ell \cdot \sum_i (x_{i,\ell} - w_{i,\ell}).$$

Substituting $w_i = \sum_j \theta_{ij} p \cdot y_j$ yields

$$p_L \cdot \sum_i \left(x_{i,L} - \sum_j \theta_{ij} p \cdot y_{j,L} \right) = -\sum_{\ell \neq L} p_\ell \cdot \sum_i \left(x_{i,\ell} - \sum_j \theta_{ij} p \cdot y_{j,\ell} \right).$$

Changing the order of summation and using $\sum_i \theta_{ij} = 1$ we get

$$p_L \cdot \left(\sum_i x_{i,L} - \sum_j y_{j,L} \right) = -\sum_{\ell \neq L} p_\ell \cdot \left(\sum_i x_{i,\ell} - \sum_j y_{j,\ell} \right) = 0,$$

and so, since $p_L \neq 0$,

$$\sum_i x_{i,L} - \sum_j y_{j,L} = 0,$$

and all the equilibrium conditions are satisfied. □

We now turn to a comparative statics example, which is taken from [1]. Suppose that a sales tax is imposed, and consumers have to pay $t > 0$ money for each unit of commodity 1 that they consume. We thus have a family of economies indexed by t, where production costs and endowments are not dependent on t, but consumer preferences are represented by

$$u_i(x, m) = u_i(x) - tx + m.$$

Following the analysis above, in equilibrium,

$$u_i'(x_i) = p + t \quad \text{and} \quad c_i'(y_i) = p. \tag{16.1}$$

How do equilibrium prices, consumption and production change when we change the tax? Denote $x = \sum_i x_i$ and $y = \sum_j y_j$ the total consumption and production in equilibrium, and by u and c the corresponding aggregate utility and cost functions. We are assume both are differentiable. Since consumption depends only on the price and tax, we denote by $x(p,t)$ consumer demand given price p and tax t. By (16.1) x depends on p and t only thought $p+t$, and thus we can write x as a function of a single variable $x(p+t) = x(p,t)$.

Production does not depend on taxation, and so we denote by $y(p)$ production at price p. We denote by $p(t)$ the equilibrium price when taxes are set to t. In equilibrium we know that

$$x(p(t)+t) = y(p(t)).$$

Differentiating this equation with respect to t yields

$$(p'(t)+1)x'(p(t)+t) = p'(t)y'(p(t)),$$

Rearranging, we get

$$-p'(t) = \frac{-x'(p(t)+t)}{y'(p(t)) - x'(p(t)+t)}.$$

We know that x' is negative and y' is positive. It thus follows that $0 \leq -p'(t) \leq 1$. That is, when we increase taxes by a cent, prices will decrease by something that is between 0 and a cent. Since consumers also pay the tax, their effective price $p(t)+t$ increases.

When $y'(p(t))$ is very big, $-p'(t)$ will be very close to 0, so that prices change very little in response to taxation. The effective price for consumers will, however, increase almost linearly with the taxation. When $y'(p(t))$ is very small, $-p'(t)$ will be almost 1. In this case the effective price to consumers will not change by much.

17 Uncertainty

In this section we explore how our setting of a private ownership economy can incorporate uncertainty. We will consider L_p *physical* commodities, and S *states of nature*. The set of commodities will have size $L = L_p \times S$. Commodities will be denoted as a pair (ℓ, s). The interpretation is that exactly one of the states realizes, and what is traded is a physical commodity whose delivery is contingent on that state. Thus, if consumer i consumes an amount $x_{i,(\ell,s)}$ of commodity (ℓ, s), that is taken to mean that contingent on the state realization s, she will get that amount of the physical commodity ℓ. Our notion of an equilibrium is unchanged, so that consumers have preferences over bundles of commodities, and an equilibrium consists of an allocation that clears the market, firms that maximize profit, and agents who consume an optimal bundle given their budget. In this setting an equilibrium is called an *Arrow-Debreu equilibrium*.

As an example, consider an exchange economy with two agents, a single physical commodity and two states. Let the preference of consumer i be given by

$$u_i(x_{i,s_1}, x_{i,s_2}) = \pi_i w_i(x_{i,s_1}) + (1-\pi_i) w_i(x_{i,s_2}),$$

for some strictly concave, differentiable $w_i : \mathbb{R} \to \mathbb{R}$ and $\pi_i \in [0,1]$. An interpretation of this preference is that the consumer is an expected utility maximizer, and has prior belief π_i that the state is s_1, and utility w_i for the physical good. Suppose that $e_1 = (1,0)$ and $e_2 = (0,1)$. Thus, in state s_1 consumer 1 is endowed with one unit of the physical product, and in state s_2 consumer 2 is endowed with one unit of the physical product.

Suppose (x_1, x_2) and p form an interior equilibrium. Then

$$\frac{p_1}{p_2} = \frac{\pi_1 w_1'(x_{1,s_1})}{(1-\pi_1) w_1'(x_{1,s_2})} = \frac{\pi_2 w_2'(x_{2,s_1})}{(1-\pi_2) w_1'(x_{2,s_2})} = \frac{\pi_2 w_2'(1-x_{1,s_1})}{(1-\pi_2) w_2'(1-x_{1,s_2})}.$$

If $\pi_1 = \pi_2 = \pi$ then this implies that

$$w_1'(x_{1,s_1}) w_2'(1-x_{1,s_2}) = w_2'(1-x_{1,s_1}) w_1'(x_{1,s_2}),$$

which is possible only if $x_{1,s_1} = x_{1,s_2}$, and hence also $x_{2,s_1} = x_{2,s_2}$, by the strict concavity of w_1 and w_2. In this case, consumers demand the same amount of physical good in both states, and thus face no risk. Prices reflect beliefs: $p_1/p_2 = \pi/(1-\pi)$. In the general case, if $\pi_1 > \pi_2$ then $x_{1,s_1} > x_{1,s_2}$ and $x_{2,s_1} < x_{2,s_2}$.

18 Pari-mutuel gambling

Consider a horse race, in which I gamblers each place bets on one of L horses. Each gambler has a budget $m_i > 0$ which they completely spend. We denote $\sum_i m_i = M$.

We will consider two equivalent ways of thinking about these markets. In the first, each gambler i decides how much money $m_{i,\ell}$ to bet on horse ℓ. After the race, if horse ℓ wins, all the money collected is distributed between those who bet on 1, in proportion to the amount they bet. So that gambler i receives

$$\frac{m_{i,\ell}}{\sum_h m_{h,\ell}} M.$$

An equivalent description is one in which, for each horse ℓ, each gambler may buy any amount of tickets for that horse, for some price p_ℓ. We denote the number of tickets bought by gambler i for horse ℓ by $x_{i,\ell}$, so that the amount spent on these tickets is $m_{i,\ell} = p_\ell x_{i,\ell}$.

After the race, given that horse ℓ won, each ticket sold for horse ℓ is worth a unit of money, and the rest are worthless. Thus, when horse ℓ wins, gambler i's payoff is equal to $x_{i,\ell}$. The market clearing condition is that the total amount of winnings distributed is always equal to the total amount of money collected. I.e., for any horse ℓ, $\sum_i x_{i,\ell} = M$. Since $x_{i,\ell} = m_{i,\ell}/p_\ell$, it follows that

$$\sum_i m_{i,\ell}/p_\ell = M,$$

or

$$p_\ell = \frac{\sum_i m_{i,\ell}}{M}.$$

That is, p_ℓ is the fraction of money gambled on horse ℓ, of all the money gambled by all the gamblers.

We assume that the gamblers each have a prior $\pi_{i,\ell}$ that horse ℓ will win the race. They are risk neutral, and so aim to maximize their expected return, which for i is equal to

$$u_i(x_i) = \sum_\ell \pi_{i,\ell} x_{i,\ell}. \tag{18.1}$$

Thus, in equilibrium, gambler i will maximize $u_i(x_i)$ subject to the constraint $\sum_\ell p_\ell x_{i,\ell} = m_i$. That is, she will gamble a positive amount on horse ℓ only if

$$\ell \in \operatorname*{argmax}_k \frac{\pi_{i,k}}{p_k}$$

A *pari-mutuel equilibrium* for priors $(\pi_{i,\ell})_{i,\ell}$ and budgets $(m_i)_i$ is a gambles profile $(x_{i,\ell})_{i,\ell}$ and a price vector p such that

1. For all i it holds that x_i maximizes $u_i(x) = \sum_\ell \pi_{i,\ell} x_{i,\ell}$ subject to $\sum_\ell p_\ell x_{i,\ell} = m_i$.

2. $\sum_i x_{i,\ell} = M$ for every ℓ.

This setting is equivalent to an exchange economy in which there is a single physical commodity (money) and L states (which horse won). Each consumer i is endowed with $e_i = (m_i, m_i, \ldots, m_i)$: a sure amount of money. Their preference is given by (18.1).

Theorem 18.1 (Eisenberg & Gale)**.** *Suppose that for each ℓ there is an i such that $\pi_{i,\ell} > 0$. Then there exist $(x_{i,\ell})_{i,\ell}$ and $p \gg 0$ that form an equilibrium.*

Note that the condition that $\pi_{i,\ell} > 0$ for some i is almost without loss of generality, for if this does not hold then we can remove horse ℓ from the race and proceed with tht rest. Before proving this theorem we will introduce *Cobb-Douglas* preferences. Consider a single consumer whose preferences over L goods is represented by the utility function

$$u(x_1, \ldots, x_L) = \prod_{\ell=1}^{L} x_\ell^{m_\ell}.$$

for some positive exponents $m_1, \ldots, m_L$. We denote $M = \sum_\ell M_\ell$. Given a price vector $p \in \mathbb{R}^L$ and total wealth w, the demand x^* will maximize u subject to $\sum_\ell p_\ell x_\ell^* \leq w$. Equivalently, we can maximize $\log u$ under this constraint. Adding a Lagrange multiplier λ for the constraint we get that x^*, λ maximize

$$\sum_\ell m_\ell \log x_\ell - \lambda(\sum_\ell p_\ell x_\ell - w).$$

The first order conditions give

$$\frac{m_\ell}{x_\ell} = \lambda p_\ell,$$

which we rearrange to

$$p_\ell x_\ell = \frac{m_\ell}{\lambda},$$

To meet the budget constraint we must have that $\lambda = w/M$, and the solution is hence

$$p_\ell x_\ell = \frac{w}{M} m_\ell.$$

An important observation is that the total amount of money spent on good ℓ is proportional to the exponent m_ℓ, regardless of the budget. This makes these preferences easy to work with, and, as we shall see now, makes them useful for other reasons.

To prove Theorem 18.1 we will consider the *social welfare* function

$$U(x) = \prod_i u_i(x_i)^{m_i}.$$

Denote

$$\Phi(x) = \log U(x) = \sum_i m_i \log u_i(x_i) = \sum_i m_i \log\left(\sum_\ell \pi_{i,\ell} x_{i,\ell}\right).$$

Note that

$$\frac{\partial \Phi}{\partial x_{i,\ell}}(x) = \frac{m_i \pi_{i,\ell}}{\sum_\ell \pi_{i,\ell} \cdot x_{i,\ell}} = \frac{m_i \pi_{i,\ell}}{u_i(x_i)}.$$

Consider the problem of maximizing U (or equivalently Φ) subject to the market clearing constraints $\sum_i x_{i,\ell} = M$ and the feasibility constraints $x_{i,\ell} \geq 0$. Clearly an optimum x exists, since U is continuous on the constrained, compact domain. In this optimum $u_i(x_i) > 0$ for all i, since otherwise $U(x) = 0$, and U takes values higher than zero, for example by having each agent gamble equal amounts on each horse.

If we denote by p_ℓ the Lagrange multiplier for the market clearing constraint $\sum_i x_{i,\ell} = M$, we get that x is a maximizer if and only if it satisfies the conditions of the KKT Theorem: $\frac{\partial \Phi}{\partial x_{i,\ell}}(x) = p_\ell$ whenever $x_{i,\ell} > 0$, in which case

$$p_\ell = \frac{m_i \pi_{i,\ell}}{u_i(x_i)}, \tag{18.2}$$

and

$$p_\ell \geq \frac{m_i \pi_{i,\ell}}{u_i(x_i)}.$$

when $x_{i,\ell} = 0$. Note that $p_\ell > 0$, since by assumption $\pi_{i,\ell} > 0$ for some i.

Multiplying both sides of (18.2) by $x_{i,\ell}$ we get

$$p_\ell x_{i,\ell} = \frac{m_i \pi_{i,\ell} x_{i,\ell}}{u_i(x_i)},$$

which holds for all ℓ: when $x_{i,\ell} > 0$ it holds by (18.2), and when $x_{i,\ell} = 0$ both sides vanish. Summing over ℓ we get

$$\sum_\ell p_\ell x_{i,\ell} = \sum_\ell \frac{m_i \pi_{i,\ell} x_{i,\ell}}{u_i(x_i)} = m_i \frac{\sum_\ell \pi_{i,\ell} x_{i,\ell}}{u_i(x_i)} = m_i,$$

since (18.2) is satisfied whenever $x_{i,\ell} \neq 0$. So miraculously the budget constraint is satisfied. This is in fact due to the clever choice of Φ as taking a Cobb-Douglas form. If $x_{i,\ell} = 0$ then by (18.2)

$$\frac{\pi_{i,\ell}}{p_\ell} \leq \frac{u_i(x)}{m_i},$$

and this holds with equality if $x_{i,\ell} > 0$. Thus $\pi_{i,\ell}/p_\ell$ is maximal for every ℓ for which $x_{i,\ell} > 0$, the choice of x_i is optimal, and $(x_{i,\ell})_{i,\ell}$ and p form an equilibrium. This completes the proof of Theorem 18.1.

While in general there is not a unique equilibrium, the prices in every equilibrium are unique. We now prove this. Suppose $(x'_{i,\ell})_{i,\ell}$ and p' form an equilibrium, as do $(x'_{i,\ell})_{i,\ell}$ and p'.

Let

$$\mu_i = \max_\ell \frac{\pi_{i,\ell}}{p_\ell}$$
$$\mu'_i = \max_\ell \frac{\pi_{i,\ell}}{p'_\ell}.$$

It follows that

$$\pi_{i,\ell} \le \mu_i p_\ell$$
$$\pi_{i,\ell} \le \mu'_i p'_\ell. \tag{18.3}$$

Under x, the expected revenue for gambler's i investment into horse ℓ is $x_{i,\ell}\pi_{i,\ell}$. By the definition of μ, $x_{i,\ell}\mu_i p_\ell = x_{i,\ell}\pi_{i,\ell}$ (when $\mu_i \ne \pi_{i,\ell}/p_\ell$ both vanish, because $x_{i,\ell}$ vanishes). Thus by (18.3)

$$x_{i,\ell}\mu_i p_\ell \le x_{i,\ell}\mu'_i p'_\ell,$$

and by a symmetric argument

$$x'_{i,k}\mu'_i p'_k \le x'_{i,k}\mu_i p_k.$$

Multiplying the inequalities by $x'_{i,k}/p_\ell$ and $x_{i,\ell}/p'_k$ respectively, we arrive at

$$x'_{i,k}x_{i,\ell}\mu_i \le x'_{i,k}x_{i,\ell}\mu'_i\frac{p'_\ell}{p_\ell}$$
$$x'_{i,k}x_{i,\ell}\mu'_i\frac{p'_k}{p_k} \le x'_{i,k}x_{i,\ell}\mu_i.$$

Combining into one inequality yields

$$x'_{i,k}x_{i,\ell}\mu'_i\frac{p'_k}{p_k} \le x'_{i,k}x_{i,\ell}\mu'_i\frac{p'_\ell}{p_\ell}.$$

We now multiply by $p'_k p_\ell$, divide by μ'_i and sum over k,ℓ:

$$\sum_{k,\ell}(p'_k x'_{i,k})(p_\ell x_{i,\ell})\frac{p'_k}{p_k} \le \sum_{k,\ell}(p'_k x'_{i,k})(p_\ell x_{i,\ell})\frac{p'_\ell}{p_\ell}.$$

Since $\sum_\ell p_\ell x_{i,\ell} = m_i = \sum_l p'_k x'_{i,k}$ we get

$$\sum_k (p'_k x'_{i,k})\frac{p'_k}{p_k} \le \sum_\ell (p_\ell x_{i,\ell})\frac{p'_\ell}{p_\ell}.$$

Summing over i and dividing by M yields

$$\sum_k p'_k \frac{p'_k}{p_k} \le \sum_\ell p_\ell \frac{p'_\ell}{p_\ell} = 1.$$

Equivalently

$$\sum_k \left(\frac{p'_k}{p_k}\right)^2 p_k \leq 1,$$

i.e., the second moment (according to p) of p'_k/p_k is at most 1. The expectation of p'_k/p_k is $\sum_k (p'_k/p_k)p_k = 1$. Since the variance of any random variable is non-negative (i.e., the second moment minus the expectation squared is non-negative) it follows that p'_k/p_k has zero variance, and thus must equal 1 for every k. Thus $p'_k = p_k$.

19 Radner equilibria

Consider an exchange economy with L_p physical goods and S states. In an Arrow-Debreu equilibrium there are prices for each of the $L = S \cdot L_p$ contingent goods. We think of trade as occurring before the state has been realized, and of consumption as occurring after the state has been realized. The equilibrium conditions are

(i) x_i maximizes $\succeq_i$ subject to $\sum_{s,\ell} p_{s,\ell} x_{i,s,\ell} \leq \sum_{s,\ell} p_{s,\ell} e_{i,s,\ell}$).

(ii) $\sum_i x_{i,s,\ell} = \sum_i e_{i,s,\ell}$ for each commodity (s,ℓ).

One could imagine that trade occurs only after the state is realized, so that the budget constraint (i) needs to hold in each state separately. This would correspond to replacing (i) by

(i') x_i maximizes $\succeq_i$ subject to $\sum_{\ell} p_{s,\ell} x_{i,s,\ell} \leq \sum_{\ell} p_{s,\ell} e_{i,s,\ell}$ for each state s.

Here, for each state s we think of each price vector $p_{s,\cdot} \in \mathbb{R}^{L_p}$ as the vector of *spot prices* for a market that opens once state s is realized. Note that if x_i satisfies (i) and (ii) then it is Pareto optimal, by the first welfare theorem. Hence there is, in a sense, no reason for further trades after the state has been revealed when all contingent goods are traded before the state is revealed. But if instead it satisfies (i') and (ii) then it may not be Pareto optimal. For example, in a market described in §17 (a single physical good, two states, $e_1 = (1,0)$ and $e_2 = (0,1)$, and expected utility maximizing consumers) it is no longer possible for the consumers to insure each other, and each must consume only in one of the states, since she has zero wealth in the other.

As we now explain, we can still achieve Pareto optimality while trading after the state is revealed, by allowing trade in just one physical good (which we can think of as money, even though we will not assume quasi-linear utilities) before the state is realized.

In a *Radner equilibrium* we consider an exchange economy with L_p physical goods and S states. We will assume that $X_i = \mathbb{R}^L_+$ and preferences are strictly monotone, and $\sum_i e_i \gg 0$. We imagine that trade takes place in two stages.

In $t = 0$ the consumers trade contingent quantities of good 1 only at some contingent price vector $q \in \mathbb{R}^S$. We denote by $z_{i,s}$ the amount of good 1 at state s that consumer i buys. Thus consumer i spends $\sum_s q_s z_{i,s}$ at $t = 0$. As none of the endowment is consumed at this stage, the consumers all have a budget of zero for this trade, and so $\sum_s q_s z_{i,s} \leq 0$. The market clearing condition is $\sum_s z_{i,s} = 0$.

In $t = 1$ the state is realized, and a spot market opens at the realized state s. The wealth of consumer i includes the value of her endowment, plus the value of the contingent goods purchased before: $\sum_{\ell} p_{s,\ell} e_{i,\ell} + p_{s,1} z_{i,s}$.

Thus, $(z_{i,s})_{i,s}$, $(x_{i,s,\ell})_{i,s,\ell}$, $(p_{s,\ell})_{s,\ell}$ and $(q_s)_s$, form a Radner equilibrium if

(i) x_i and z_i maximize $\succeq_i$ subject to $\sum_s q_s z_{i,s} \leq 0$ and $\sum_{\ell} p_{s,\ell} x_{i,s,\ell} \leq \sum_{\ell} p_{s,\ell} e_{i,s,\ell} + p_{s,1} z_{i,s}$ for each state s.

(ii) $\sum_i z_{i,s} = 0$ for each state s.

(iii) $\sum_i x_{i,s,\ell} = \sum_i e_{i,s,\ell}$ for each commodity (s,ℓ).

Note that when trading at $t=0$, consumers are (correctly) anticipating the prices at $t=1$, basing their choice of z_i on them. The next claim shows that we can assume without loss of generality that $q_s = p_{s,1}$, so that the price of good 1 does not change from $t=0$ to $t=1$.

Claim 19.1. *Suppose $(z_{i,s})_{i,s}$, $(x_{i,s,\ell})_{i,s,\ell}$, $(p_{s,\ell})_{s,\ell}$ and $(q_s)_s$ form a Radner equilibrium, with $p_{s,1}>0$. Let $p'_{s,\ell} = \frac{q_s}{p_{s,1}}p_{s,\ell}$ (so that $q_s = p'_{s,1}$). Then $(z_{i,s})_{i,s}$, $(x_{i,s,\ell})_{i,s,\ell}$, $(p'_{s,\ell})_{s,\ell}$ and $(q_s)_s$ form a Radner equilibrium.*

Proof. It suffices to check that x_i and z_i maximize $\succeq_i$ subject to $\sum_\ell p'_{s,\ell}(x_{i,s,\ell}-e_{i,s,\ell}) \le p'_{s,1}z'_{i,s}$ and $\sum_s q_s z_{i,s} \le 0$ for each state s for each state s.

This indeed holds, since the condition

$$\sum_\ell p'_{s,\ell}(x_{i,s,\ell}-e_{i,s,\ell}) \le p'_{s,1}z_{i,s}$$

is equivalent to

$$\sum_\ell \frac{q_s}{p_{s,1}}p_{s,\ell}(x_{i,s,\ell}-e_{i,s,\ell}) \le \frac{q_s}{p_{s,1}}p_{s,1}z_{i,s},$$

which is equivalent to

$$\sum_\ell p_{s,\ell}(x_{i,s,\ell}-e_{i,s,\ell}) \le p_{s,1}z_{i,s}.$$

□

Note that since $\sum_i e_i \gg 0$, since preferences are strictly monotone, prices must be positive, since otherwise maximal demands would not exist. We will therefore describe Radner equilibria by just specifying z, x and p, with the implied assumption that $q_s = p_{s,1}$.

Proposition 19.2.
1. *Suppose $(z_{i,s})_{i,s}$, $(x_{i,s,\ell})_{i,s,\ell}$ and $(p_{s,\ell})_{s,\ell}$ form a Radner equilibrium. Then $(x_{i,s,\ell})_{i,s,\ell}$ and $(p_{s,\ell})_{s,\ell}$ form an Arrow-Debreu equilibrium.*
2. *Suppose $(x_{i,s,\ell})_{i,s,\ell}$, and $(p_{s,\ell})_{s,\ell}$ form an Arrow-Debreu equilibrium. Then there are $(z_{i,s})_{i,s}$ such that $(z_{i,s})_{i,s}$, $(x_{i,s,\ell})_{i,s,\ell}$ and $(p_{s,\ell})_{s,\ell}$ form a Radner equilibrium.*

Proof. In the Arrow-Debreu setting, consumer i chooses x_i as a maximal element from

$$A_i = \left\{(x_{i,s,\ell})_{s,\ell} : \sum_{s,\ell} p_{s,\ell}(x_{i,s,\ell}-e_{i,s,\ell}) \le 0\right\}$$

We can reformulate the consumer's problem in the the Radner setting to maximize x_i over the set

$$R_i = \left\{(x_{i,s,\ell})_{s,\ell} : \exists (z_{i,s})_s \text{ s.t. } \sum_s p_{s,1}z_{i,s} \le 0 \text{ and } \sum_\ell p_{s,\ell}(x_{i,s,\ell}-e_{i,s,\ell}) \le p_{s,1}z_{i,s} \text{ for all } s\right\}$$

We show that $A_i = R_i$. Fix $x_i \in A_i$. Let

$$z_{i,s} = \frac{1}{p_{s,1}} \sum_\ell p_{s,\ell}(x_{i,s,\ell} - e_{i,s,\ell}). \tag{19.1}$$

Then it is immediate that $\sum_s p_{s,1} z_{i,s} \le 0$ and $\sum_\ell p_{s,\ell}(x_{i,s,\ell} - e_{i,s,\ell}) \le p_{s,1} z_{i,s}$ (the latter with equality), and so $x_i \in R_i$.

Conversely, fix $x_i \in R_i$. Then summing the second inequality in the definition of R_i over s yields

$$\sum_{s,\ell} p_{s,\ell}(x_{i,s,\ell} - e_{i,s,\ell}) \le \sum_s p_{s,1} z_{i,s},$$

which is at most 0 by the first inequality in the definition of R_i. Thus $x_i \in A_i$.

1. Since $R_i = A_i$, x_i is a maximal element of A_i. Since it is also an allocation, we have an Arrow-Debreu equilibrium.

2. Since $R_i = A_i$, x_i is a maximal element of R_i, and furthermore, as the proof above shows, if we choose z_i by (19.1), x_i and z_i will solve the consumer's problem. Since x_i is an allocation, it remains to be shown that $\sum_i z_{i,s} = 0$ for all s. By our choice of $z_{i,s}$,

$$\sum_i z_{i,s} = \sum_i \frac{1}{p_{s,1}} \sum_\ell p_{s,\ell}(x_{i,s,\ell} - e_{i,s,\ell}) = \frac{1}{p_{s,1}} \sum_\ell p_{s,\ell} \sum_i (x_{i,s,\ell} - e_{i,s,\ell})$$

   Since $(x_{i,s,\ell})_i$ is an allocation, $\sum_i (x_{i,s,\ell} - e_{i,s,\ell}) = 0$, and so $\sum_i z_{i,s} = 0$.

□

20 Asset pricing

We will now consider an economy with a single physical commodity (potatoes) and a finite set of S states. Consumers have strictly monotone preferences over the S contingent commodities. Each consumer is endowed with some $e \in \mathbb{R}^S_+$. An *asset* $a \in \mathbb{R}^S$ is a contract that delivers a_s potatoes in state s. We will think of assets as column vectors.

Some examples of assets include the risk free asset $a_s = 1$. Another is the for state s', given by $a_s = 1_{s=s'}$. Given an asset a, an asset b is a *call option* for a with *strike price* p if $b_s = (a_s - p)^+$. The asset c is a *put option* with strike price p if $c_s = (p - a_s)^+$.

A market will consist of J assets $a^1, \ldots, a^J$, and an asset price vector $q \in \mathbb{R}^J_+$, which we will think of as also quoted in potatoes. We will denote by A the matrix whose columns are the a^j's, so that $A_{s,j} = a^j_s$. A *portfolio* for A is $z \in \mathbb{R}^J$. The price of a portfolio z is $q \cdot z$. It generates the cash (or potato) flow Az: in state s, portfolio z delivers $\sum_j A_{s,j} z_j$ potatoes.

Given a market with assets A and prices q, a consumer with endowment e will choose a portfolio $z \in \mathbb{R}^J$ to maximize her consumption $x = e + Az$, subject to $q \cdot z \le 0$.

We say that a portfolio z is an *arbitrage opportunity* if $q \cdot z \le 0$ and $Az \ge 0$, $Az \ne 0$. Clearly, if there are arbitrage opportunities in the market then there is no solution to the consumer's problem, and any notion of equilibrium is precluded.

Theorem 20.1. *Suppose that $A_{s,j} \ge 0$, and that for all j there is an s such that $A_{s,j} > 0$. Then the following are equivalent:*

(i) There are no arbitrage opportunities.

(ii) There is a $\mu \gg 0 \in \mathbb{R}^S$ such that $\mu A = q$.

Proof. Assume (i). We note first that $q \gg 0$, since if $q_j \le 0$ then z with $z_j = 1$, $z_k = 0$ for $k \ne j$ is an arbitrage opportunity. Consider the set

$$V = \{Az \in \mathbb{R}^S : q \cdot z \le 0\}.$$

This convex subset of $\mathbb{R}^S$ consists of all flows that can be achieved with a balanced budget portfolio, and thus $V \cap \mathbb{R}^S_+ = \{0\}$. Hence, of we denote $\Delta^S = \{x \in \mathbb{R}^S_+ : \sum_s x_s = 1\}$, then $V \cap \Delta^S$ is empty. Since Δ^S is compact, by the separating hyperplane theorem there is a $\mu \ne 0$ such that $\mu \cdot x > 0$ for all $x \in \Delta^S$ and $\mu \cdot v \le 0$ for all $v \in V$. By the first property of μ we have that $\mu \gg 0$. The second property implies that $\mu \cdot v = 0$ for all $v \in V_0 = \{Az \in \mathbb{R}^S : q \cdot z = 0\}$, since $V_0 \subseteq V$ is a vector space. We thus have that $\mu A z = 0$ whenever $q \cdot z = 0$. So μA is orthogonal to every vector in the space of vectors orthogonal to q, and hence must be equal to q, up to some constant λ. Since $\mu \gg 0$ and by our assumptions on A, $\mu A \gg 0$. Since $q \in \mathbb{R}^S_+$, λ must be positive, and, by rescaling μ, we have that $q = \mu A$.

Now suppose that (i) does not hold, so that that there is a portfolio z with $q \cdot z \le 0$ and $Az \in \Delta^S$. Then for any $\mu \gg 0$ it holds that $\mu A \gg 0$, and so

$$\mu A z = (\mu A) \cdot z > 0.$$

□

An implication of this result is that when there are no arbitrage opportunities, the price q_j of asset a^j is

$$q_j = \mu \cdot a^j = \sum_s \mu_s a^j_s.$$

So the price of every asset is a weighted average of what it delivers in the different states. As we note in the proof of the theorem, under the positivity assumptions on A, q is positive. Since the consumers' problem is unchanged when q is multiplied by a constant, we can assume that μ is in Δ^S. So we can interpret it as a probability measure on the set of states, and hence each asset costs the expectation of what it delivers in the different states, according to this distribution.

We say that A is complete if it has rank S.

Theorem 20.2. *Suppose that $A_{s,j} \geq 0$, that for all j there is an s such that $A_{s,j} > 0$. Then the following are equivalent:*

(i) The market is complete.

(ii) There is a unique $\mu \in \mathbb{R}^S$ such that $\mu A = q$.

Proof. Suppose that the market is complete, so that A has full rank. Then $\nu A = 0$ implies $\nu = 0$. It follows that if $\mu A = q$ and $\mu' A = q$ then $(\mu - \mu')A = 0$, and thus $\mu = \mu'$. Conversely, if A does not have full rank, then there is some $\nu \neq 0$ such that $\nu A = 0$. Hence if $\mu A = q$ then also $(\mu + \nu)A = q$, and μ is not unique. □

Index

allocation, 8
approximate equilibrium, 18
arbitrage opportunity, 42
Arrow-Debreu equilibrium, 33
Arrow-Debreu security, 42
asset, 42

call option, 42
coalition, 22
Cobb-Douglas, 35
consumer surplus, 29
convex*, 4
core, 22
correspondence, 16

dense, 4

excess demand, 13

free disposal, 14

homogeneous of degree zero, 13

indirect utility, 7

locally non-satiated, 4
lower contour set, 4

Minkowski sum, 7

Pareto optimal, 8
pari-mutuel equilibrium, 34
portfolio, 42
production set, 7
put option, 42

quasiconcave, 4

Radner equilibrium, 39

Scitovsky contour, 21
separable, 4
spot prices, 39
strictly monotone, 22
strike price, 42
support function, 7

upper contour set, 4
upper hemicontinuous, 16

Walras's Law, 13
Walrasian equilibrium, 8
Walrasian equilibrium with transfers, 10
Walrasian quasi-equilibrium with transfers, 10

References

[1] A. Mas-Colell, M. D. Whinston, J. R. Green, et al. *Microeconomic theory*, volume 1. Oxford university press New York, 1995.